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Flow-contractible configurations and group connectivity of signed graphs

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ABSTRACT

Jaeger, Linial, Payan and Tarsi (JCTB, 1992) introduced the concept of group connectivity as a generalization of nowhere-zero flow for graphs. In this paper, we introduce group connectivity for signed graphs and establish some fundamental properties. For a finite abelian group A, it is proved that an A-connected signed graph is a contractible configuration for A-flow problem of signed graphs. In addition, we give sufficient edge connectivity conditions for signed graphs to be A-connected and study the group connectivity of some families of signed graphs.

1. Introduction

The notion of nowhere-zero flows of ordinary graphs was introduced by Tutte [15,16] as a dual problem to vertex coloring of graphs embedded on an orientable surface. The definition of nowhere-zero flows of signed graphs naturally comes from the study of embeddings of graphs in non-orientable surfaces, where nowhere-zero flows emerge as the dual notion to local tensions.

The group connectivity, as a generalization of the flow problem, is a concept introduced by Jaeger, Linial, Payan and Tarsi [5]. Furthermore, graphs with certain group connectivity are contractible configurations for flow problems.

In this paper, the concept and results about group connectivity [5] for ordinary graphs are extended to signed graphs.

1.1. *Group connectivity for ordinary graphs*

Throughout the paper, we consider finite graphs. Loops and multiple edges are allowed. We refer [21] for undefined notations and terminology on nowhere-zero flows.

Let A be a non-trivial (additive) abelian group with additive identity 0, and let $A^* = A\setminus\{0\}$ be the set of nonzero elements in A. Let D be an orientation of G. Define $F(G, A) = \{f \,|\, f : E(G) \mapsto A\}$ and $F^*(G, A) = \{f \,|\, f : E(G) \mapsto A^*\}$. For each $f \in F(G, A)$, the *boundary* of f is the function $\partial f : V(G) \mapsto A$ defined by $\partial f(v) = \sum_{e\in E^+_D(v)} f(e) - \sum_{e\in E^-_D(v)} f(e)$ for each vertex $v \in V(G)$. (D, f) is called an *A-flow* if $\partial f = 0$, and is called a *nowhere-zero A-flow* if moreover $f \in F^*(G, A)$. If $A = \mathbb{Z}$ and $1 \le |f(e)| \le k - 1$ for each $e \in E(G)$, the flow (D, f) is called a *nowhere-zero k-flow*. Tutte's flow conjectures are some of the major open problems in graph theory. The 3-flow conjecture states that *every* 4-*edge-connected graph admits a nowhere-zero* 3-*flow* and the 5-flow

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conjecture claims that *every bridgeless graph admits a nowhere-zero* 5*-flow*. The readers are referred to [9] for a recent survey on this topic.

Jaeger, Linial, Payan and Tarsi [5] introduced the concept of *group connectivity* as a generalization of nowhere-zero flows of graphs. It is obvious that $\sum_{v\in V(G)}\partial f(v) = 0$ for any $f \in F^*(G, A)$. This motivates the definition of *A-boundary function*. A mapping $b : V(G) \mapsto A$ is called an *A-boundary* of G if $\sum_{v\in V(G)} b(v) = 0$. Let $Z(G, A)$ be the collection of all A-boundaries of G. G is *A-connected* if, for any $b \in Z(G, A)$, there is a function $f \in F^*(G, A)$ such that $\partial f = b$, that is, for every vertex $v \in V(G)$,

$$\partial f(v) = \sum_{e\in E_D^+(v)} f(e) - \sum_{e\in E_D^-(v)} f(e) = b(v).$$

Jaeger et al. [5] conjectured that *every* 5*-edge-connected graph is* $\mathbb{Z}_3$*-connected,* and *every* 3*-edge-connected graph is* $\mathbb{Z}_5$*-connected*. These two conjectures imply Tutte's 3-flow conjecture and 5-flow conjecture, respectively. Jaeger et al. [5] proved that every 4-edge-connected graph is A-connected for any abelian group A with $|A| \geq 4$. Thomassen's breakthrough result in [14] confirmed the conjecture of Jaeger et al. for 8-edge-connected graphs, and it was later improved by Lovász et al. [10] that *every* 6*-edge-connected graph is* $\mathbb{Z}_3$*-connected.* In this paper, we will introduce the concept of group connectivity for signed graphs and extend the above mentioned results to signed graphs with slightly higher edge-connectivity.

1.2. Preliminary for signed graphs

A *signed graph* is a graph G with a mapping $\sigma : E(G) \mapsto \{1, -1\}$. An edge $e \in E(G)$ is positive if $\sigma(e) = 1$ and negative if $\sigma(e) = -1$. The mapping σ, called signature, is sometimes implicit in the notation of a signed graph and will be specified when needed. Both negative and positive loops are allowed in signed graphs, while positive loops do not affect any flow property. We use $E_\sigma^+(G)$ and $E_\sigma^-(G)$ to denote the set of positive edges and the set of negative edges in G, respectively. If no confusion occurs, we simply use E_σ^+ for $E_\sigma^+(G)$ and E_σ^- for $E_\sigma^-(G)$. An *orientation* τ assigns each edge of (G, σ) as follows: if $e = xy$ is a positive edge, then the edge is either oriented away from x and toward y or away from y and toward x; if $e = xy$ is a negative edge, then the edge is oriented either away from both x and y or towards both x and y. We call $e = xy$ a *sink edge* (a *source edge*, respectively) if it is oriented away from (towards, respectively) both x and y.

Let τ be an orientation of (G, σ). For each vertex $v \in V(G)$, let $H_G(v)$ be the set of half edges incident with v. Define $\tau(h) = 1$ if the half edge $h \in H_G(v)$ is oriented away from v, and $\tau(h) = -1$ if the half edge $h \in H_G(v)$ is oriented towards v. Denote $d_\tau^+(v) = |E_\tau^+(v)|$ ($d_\tau^-(v) = |E_\tau^-(v)|$, respectively) to be the outdegree (indegree, respectively) of (G, σ) under orientation τ, where $E_\tau^+(v)$($E_\tau^-(v)$, respectively) denotes the set of outgoing (ingoing, respectively) half edges incident with v.

The *switch operation* ζ on an edge-cut S is a mapping $\zeta : E(G) \mapsto \{-1, 1\}$ such that $\zeta(e) = -1$ if $e \in S$ and $\zeta(e) = 1$ otherwise. Two signatures σ and σ' are *equivalent* if there exists an edge-cut S such that $\sigma(e) = \sigma'(e)\zeta(e)$ for every edge $e \in E(G)$, where ζ is the switch operation on the edge-cut S. For a signed graph (G, σ), let $\mathcal{X}$ denote the collection of all signatures equivalent to σ. The *negativeness* of (G, σ) is denoted by $\epsilon_N(G, \sigma) = \min\{|E_{\sigma'}^-(G)| : \forall\sigma' \in \mathcal{X}\}$. We use ϵ_N for short if the signed graph (G, σ) is understood from the context. A signed graph is called *k-unbalanced* if $\epsilon_N \geq k$. Note that 1-unbalanced signed graph is also known as unbalanced signed graph.

A circuit is *balanced* if $\epsilon_N = 0$ and is *unbalanced* otherwise (i.e. $\epsilon_N = 1$). A signed graph (G, σ) is called a barbell if either

- G consists of two unbalanced circuits C_1, C_2 with $|V(C_1) \cap V(C_2)| = 1$, or
- G consists of two vertex disjoint unbalanced circuits C_1, C_2 and a path P, which has one end in $V(C_1)$ and one end in $V(C_2)$ and has no interior vertices in $V(C_1) \cup V(C_2)$.

A *signed circuit* is either a balanced circuit or a barbell.

The signature is usually implicit in the notation of a signed graph if no confusion occurs. We define *contraction* in signed graphs as follows. For an edge $e \in E(G)$, the *contraction* G/e is the signed graph obtained from G by identifying the two ends of e, and then deleting the resulting positive loop if $e \in E_\sigma^+$, but keeping the resulting negative loop if $e \in E_\sigma^-$, For $X \subseteq E(G)$, the *contraction* G/X is the signed graph obtained from G by contracting all edges in X. If H is a subgraph of G, we use G/H for $G/E(H)$. An immediate observation is that the contraction operation does not decrease negativeness. That is, $\epsilon_N(G/H) \geq \epsilon_N(G)$ for any subgraph H of G.

1.3. Group connectivity of signed graphs

Let A be an abelian group, $2A = \{2\alpha : \alpha \in A\}$, and $A^* = A \setminus \{0\}$. For a signed graph G, we still denote $F(G, A) = \{f | f : E(G) \mapsto A\}$ and $F^*(G, A) = \{f | f : E(G) \mapsto A^*\}$. Let τ be an orientation of (G, σ). For each $f \in F(G, A)$, the *boundary* of f is the function $\partial f : V(G) \mapsto A$ defined by

$$\partial f(v) = \sum_{h\in H_G(v)} \tau(h)f(e_h),$$

where e_h is the edge of G containing h and "$\sum$" refers to the addition in A. If $\partial f = 0$, then (τ, f) is called an *A-flow* of G. In addition, (τ, f) is a *nowhere-zero A-flow* if $f \in F^*(G, A)$ and $\partial f = 0$.