Discrete Mathematics 341 (2018) 3249–3270

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The pinnacle set of a permutation

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ARTICLE INFO

Article history:
Received 15 May 2017
Received in revised form 7 August 2018
Accepted 9 August 2018

Keywords:
Peak set
Descent set
Permutation statistic

ABSTRACT

The peak set of a permutation records the indices of its peaks. These sets have been studied in a variety of contexts, including recent work by Billey, Burdzy, and Sagan, which enumerated permutations with prescribed peak sets. In this article, we look at a natural analogue of the peak set of a permutation, instead recording the values of the peaks. We define the "pinnacle set" of a permutation w to be the set $\{w(i) : i \text{ is a peak of } w\}$. Although peak sets and pinnacle sets mark the same phenomenon for a given permutation, the behaviors of these sets differ in notable ways as distributions over the symmetric group. In the work below, we characterize admissible pinnacle sets and study various enumerative questions related to these objects.

1. Introduction

Let S_n denote the set of permutations of $[n] = \{1, 2, \ldots, n\}$, which we will always write as words, $w = w(1)w(2)\cdots w(n)$. An *ascent* of a permutation w is an index i such that $w(i) < w(i+1)$, while a *descent* is an index i such that $w(i) > w(i+1)$. A *peak* is a descent that is preceded by an ascent, whereas a *valley* is an ascent that is preceded by a descent. This terminology refers to the shape of the graph of w, that is, the set of points $(i, w(i))$.

Example 1.1. The descents of $315264 \in S_6$ are 1, 3, and 5, and the ascents are 2 and 4. The peaks are 3 and 5, while the valleys are 2 and 4.

The *descent set* of a permutation w, denoted $\mathrm{Des}(w)$, is the collection of its descents,

$$\mathrm{Des}(w) = \{i : w(i) > w(i+1)\} \subseteq [n-1],$$

while the *peak set* of a permutation w, denoted $\mathrm{Pk}(w)$, is the collection of its peaks,

$$\mathrm{Pk}(w) = \{i : w(i-1) < w(i) > w(i+1)\} \subseteq \{2, 3, \ldots, n-1\}.$$

Note in particular that the descent set completely determines the peak set:

$$\mathrm{Pk}(w) = \{i > 1 : i \in \mathrm{Des}(w) \text{ and } i-1 \notin \mathrm{Des}(w)\}.$$

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https://doi.org/10.1016/j.disc.2018.08.011

Any subset of $\{1, 2, \ldots, n-1\}$ is the descent set of some permutation in S_n, but the same cannot be said for peak sets. First of all, peaks cannot occur in the first or last positions of a permutation, so $\mathrm{Pk}(w) \subseteq \{2, \ldots, n-1\}$ for any $w \in S_n$. Moreover, peaks cannot occur in consecutive positions, so if $i \in \mathrm{Pk}(w)$ then $i \pm 1 \notin \mathrm{Pk}(w)$. This characterization of peak sets, as subsets of $\{2, \ldots, n-1\}$ with no consecutive elements, turns out to imply that the number of distinct peaks sets is given by the Fibonacci numbers.

It has long been known that counting permutations according to the number of descents gives rise to the Eulerian numbers, while the number of permutations with a given descent set is also well known; see, e.g., [20, Example 2.2.4]. More recently Billey, Burdzy, and Sagan [3] considered the related enumerative question for peaks: how many permutations in S_n have a given peak set? One of their results is that for a fixed set S, the number of $w \in S_n$ for which $\mathrm{Pk}(w) = S$ is a power of two times a polynomial in n, and they give techniques for explicit computation of this polynomial in special cases. As a follow up to this work, Kasraoui [14] verified their related conjecture about which peak sets of a given cardinality maximize the number of permutations in S_n for a given n.

In the present article, we study analogous questions related to peaks, but rather than tracking peaks by their positions (x-coordinates in the graph of the permutation), we use their values (y-coordinates).

Definition 1.2. A *pinnacle* of a permutation w is a value $w(i)$ such that $w(i-1) < w(i) > w(i+1)$; equivalently, j is a pinnacle of w if and only if $w^{-1}(j) \in \mathrm{Pk}(w)$. The *pinnacle set* of w is

$$\mathrm{Pin}(w) = \{w(i) : i \in \mathrm{Pk}(w)\}.$$

Certainly $|\mathrm{Pk}(w)| = |\mathrm{Pin}(w)|$, but the sets themselves need not be the same, as we now demonstrate.

Example 1.3. If $w = 315264$, then $\mathrm{Pk}(w) = \{3, 5\}$ and $\mathrm{Pin}(w) = \{5, 6\}$.

The definition of pinnacle sets leads naturally to questions about the value

$$p_S(n) := |\{w \in S_n : \mathrm{Pin}(w) = S\}|. \tag{1}$$

While similar notation was used to denote the peak polynomial, e.g. in [3,4,7], note that $p_S(n)$ is counting the number of permutations with a given pinnacle set S in this paper. The questions we address in this article are the following.

Question 1.4. *When is $p_S(n) > 0$? That is, which sets S are the pinnacle set of some permutation in S_n?*

Question 1.5. *Given a pinnacle set $S \subseteq [n]$, how do we compute $p_S(n)$?*

Question 1.6. *For a given n, what choice of $S \subseteq [n]$ maximizes or minimizes $p_S(n)$?*

In Section 2 we identify conditions under which a set S is the pinnacle set for some permutation, fully answering Question 1.4.

Definition 1.7. A set S is an *n-admissible pinnacle set* if there exists a permutation $w \in S_n$ such that $\mathrm{Pin}(w) = S$. If S is n-admissible for some n, then we simply say that S is *admissible*.

The empty set is always an n-admissible pinnacle set, because it is the pinnacle set of the identity permutation. Examples of nonempty admissible pinnacle sets are shown in Table 1. The main result about admissible pinnacle sets is the following.

Theorem 1.8 (*Admissible Pinnacle Sets*)**.** *Let S be a nonempty set of integers with $\max S = m$. Then S is an admissible pinnacle set if and only if both*

1. *$S \setminus \{m\}$ is an admissible pinnacle set, and*
2. *$m > 2|S|$.*

Moreover, there are $\binom{m-2}{\lfloor m/2 \rfloor}$ admissible pinnacle sets with maximum m, and

$$1 + \sum_{m=3}^{n} \binom{m-2}{\lfloor m/2 \rfloor} = \binom{n-1}{\lfloor (n-1)/2 \rfloor},$$

admissible pinnacle sets $S \subseteq [n]$.

Our characterization of admissible pinnacle sets is in contrast to the characterization of peak sets mentioned earlier. Whereas the number of peak sets is given by the Fibonacci numbers, here we get a central binomial coefficient.

In Section 3 we develop both a quadratic and a linear recurrence for $p_S(n)$, which partially answers Question 1.5. Further, we identify the following bounds for $p_S(n)$ partially answering Question 1.6; the sets which achieve the tight bounds are constructed in Section 3.3.