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Grinberg's Criterion

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ABSTRACT

We generalize Grinberg's hamiltonicity criterion for planar graphs. To this end, we first prove a technical theorem for embedded graphs. As a special case of a corollary of this theorem we obtain Zaks' extension of Grinberg's Criterion (which encompasses earlier work of Gehner and Shimamoto), but the result also implies Grinberg's formula in its original form in a much broader context. Further implications are a short proof for a slightly strengthened criterion of Lewis bounding the length of a shortest closed walk from below as well as a generalization of a theorem due to Bondy and Häggkvist.

1. Introduction

This paper extends what has been referred to in the literature as the Kozyrev–Grinberg method, see for instance Tutte [19], Honsberger [10, Chapter 7] or Sachs' original source [15]. However, as Honsberger writes, the mathematical discovery seems to belong to Grinberg, while Kozyrev helped to make the result well known. We therefore will use only Grinberg's name in the following.

All embeddings in this paper are assumed to be 2-cell embeddings on orientable surfaces. We assume the embeddings to be given by an orientation of the edges around each vertex. If we have a subgraph of an embedded graph, the embedding of the subgraph is induced from the orientation around the vertices, so this induced embedding can be in a surface of smaller genus. With Steinitz' Theorem in mind, we call plane 3-connected graphs *polyhedral*. In 1880 Tait conjectured that every cubic polyhedral graph is hamiltonian. Had this conjecture been true, it would have implied the Four Colour Theorem. For historical details and references, we refer to [2]. Tutte [18] was the first to give a counterexample to Tait's conjecture in 1946. In 1968 Grinberg [8] published a necessary condition

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– which we will refer to as Grinberg's Criterion – for a planar graph to be hamiltonian (given in detail in Section 2). Since its inception, it has turned out to be one of the few powerful tools at our disposal to ascertain that a given planar graph is non-hamiltonian. Following Bondy and Murty [4], Kirkman [12] was aware of the identity behind the criterion already in 1881, but Kirkman used it to search for hamiltonian cycles in certain cubic polyhedral graphs, of which he was convinced that they are hamiltonian.

For classic sources on the criterion, consult [1,10], and for examples of recent related work, see [5,11,13,21]. Grinberg himself used it to construct cubic polyhedral graphs that provide smaller (than Tutte's) counterexamples to Tait's conjecture. The non-hamiltonicity of Tutte's graph can also be proven with the criterion, see Honsberger's account of a proof due to Watts [10, pp. 86–88] or West [20, pp. 303–304]. West provides in his book further applications of the criterion, for instance details on how a similar approach to the one used for Tutte's graph leads to a short proof of the fact that the smallest counterexample to Tait's conjecture, the Lederberg–Bosák–Barnette graph on 38 vertices [9], is indeed non-hamiltonian.

Gehner [6] and Shimamoto [16] extended the criterion in two different (natural) directions, while Zaks [22] unified the two approaches. All of these deal with plane graphs. Here we extend Zaks' version of the criterion in various ways and show that even Grinberg's original equation holds under much weaker assumptions. We also discuss some applications, including a theorem of Lewis [13], as well as a result due to Bondy and Häggkvist [3].

To our knowledge, to date Grinberg's Criterion has been applied only in the context of planar graphs (see e.g. Thomassen's paper [17] or recent work of Jooyandeh et al. [11]), albeit sometimes to prove a result on non-planar graphs. Applications of the latter type include Chia and Thomassen's recent paper [5] in which a unified proof of independent results of Robertson, Bondy, Thomason, and Schwenk about the number of hamiltonian cycles in generalized Petersen graphs is given as well as an article by Wiener [21] in which he constructs the smallest known hypohamiltonian graph with crossing number 1.

2. A technical result

We recall Grinberg's theorem and then present a technical generalization of the theorem. Evaluating the cases when some of the parameters cancel out will result in corollaries that are – though just special cases of the general theorem – results that are interesting in themselves.

Theorem 2.1 (*Grinberg's Criterion (Grinberg, 1968 [8])*). *Given a plane graph with a hamiltonian cycle S and f_k (f'_k) faces of size k inside (outside) of S, we have*

$$\sum_{k\geq 3}(k-2)(f_k-f'_k)=0.$$

Let $G=(V,E)$ be an embedded graph and $S=(V_S,E_S)$ be a subgraph of G. Let $G_d=(V_d,E_d)$ be the dual graph of G and for $e\in E$ let e_d denote the corresponding edge in E_d. For a face f in G the corresponding vertex in G_d is f_d. Let $C_{d,1},\ldots,C_{d,k}$ be the components of $(V_d, E_d\setminus\{e_d|e\in E_S\})$. For each such component $C_{d,i}$ we define the (embedded) component $C_i=(V_i,E_i)$ under decomposition by S with V_i, respectively E_i the set of all vertices, respectively edges, contained in a face f so that $f_d\in C_{d,i}$. The rotational order around the vertices is induced by G. Faces that correspond to a vertex $f_d\in C_{d,i}$ are called *internal* faces, the others are called *external*. The genus of an embedded graph G, that is, the genus of the surface it is embedded in, is denoted by $\gamma(G)$, and for a face f its size (that is: the length of the facial walk) is denoted by $s(f)$.

The *decomposition graph* (possibly with loops) $D_{G,S}=(V_{G,S},E_{G,S})$ of G by S is defined as $V_{G,S}=\{C_1,\ldots,C_k\}$ and $\{C_i,C_j\}\in E_{G,S}$ if and only if there is an edge of E_S with a face of C_i on one side and a face of C_j on the other. To illustrate the construction of $D_{G,S}$, one can consider the setting of Grinberg's original theorem, that is, a plane graph G with a hamiltonian cycle S. Then there are two components