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Johnson's bijections and their application to counting simultaneous core partitions

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ABSTRACT

Johnson recently proved Armstrong's conjecture which states that the average size of an (a, b)-core partition is $(a + b + 1)(a - 1)(b - 1)/24$. He used various coordinate changes and one-to-one correspondences that are useful for counting problems about simultaneous core partitions. We give an expression for the number of $(b_1, b_2, \dots, b_n)$-core partitions where $\{b_1, b_2, \dots, b_n\}$ contains at least one pair of relatively prime numbers. We also evaluate the largest size of a self-conjugate $(s, s + 1, s + 2)$-core partition.

1. Introduction

Let $\mathbb{N}$ denote the set of non-negative integers. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is an ℓ-tuple of non-increasing positive integers with $\sum_{i=1}^{\ell}\lambda_i = n$, then we call λ a *partition* of n. One can visualize λ by using *Ferrers diagram* as in Fig. 1. Each square in a Ferrers diagram is called a *cell*. By counting the number of cells in its NE (North East) and NW (North West) direction including itself, we define the *hook length* of a cell. For example, the hook length of the colored cell in Fig. 1 is 6.

We say λ is an *a-core partition* (or, simply an *a-core*) if there is no cell whose hook length is divisible by a. Similarly, we say a partition is an $(a_1, a_2, \dots, a_n)$*-core* if it is simultaneously an a_1-core, an a_2-core, $\dots$, and an a_n-core.

Anderson [4] proved that if a and b are coprime, the number of (a, b)-cores is $\mathrm{Cat}_{a,b} := \frac{1}{a+b}\binom{a+b}{a}$, which is a generalized Catalan number. Since Anderson [4], many mathematicians have been conducting research on counting simultaneous core partitions and related subjects: [1–3,5,8–11,14–16,18].

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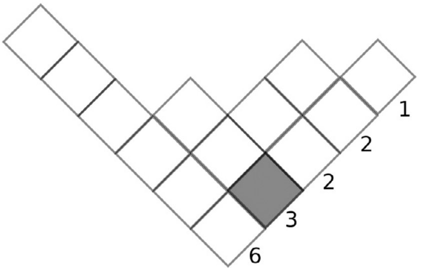

Fig. 1. $\lambda = (6, 3, 2, 2, 1)$.

Armstrong [5] conjectured that if a and b are coprime, the average size of an (a, b)-core partition is $(a + b + 1)(a - 1)(b - 1)/24$. Johnson [8] recently proved Armstrong's conjecture by using Ehrhart theory. A proof without Ehrhart theory was given by Wang [13].

In [8], Johnson established a bijection between the set of (a, b)-cores and the set

$$\left\{(z_0, z_1, \dots, z_{a-1}) \in \mathbb{N}^a : \sum_{i=0}^{a-1} z_i = b \text{ and } a \mid \sum_{i=0}^{a-1} iz_i \right\}.$$

By showing that the cardinality of this set is $\mathrm{Cat}_{a,b}$, he gave a new proof of Anderson's theorem. Inspired by Johnson's method and this bijection, we count the number of simultaneous core partitions. We find a general expression for the number of $(b_1, b_2, \dots, b_n)$-core partitions where $\{b_1, b_2, \dots, b_n\}$ contains at least one pair of relatively prime numbers. As a corollary, we obtain an alternative proof for the number of $(s, s+d, s+2d)$-core partitions, which was given by Yang–Zhong–Zhou [17] and Wang [13]. Subsequently, we also give a formula for the number of $(s, s + d, s + 2d, s + 3d)$-core partitions.

Many authors have studied core partitions satisfying additional restrictions. For example, Berg and Vazirani [7] gave a formula for the number of a-core partitions with largest part x. We generalize this formula, giving a formula for the number of a-core partitions with largest part x and second largest part y.

This paper also includes a result related to the largest size of a simultaneous core partition which has been studied by many mathematicians. For example, Aukerman, Kane and Sze [6, Conjecture 8.1] conjectured that if a and b are coprime, the largest size of an (a, b)-core partition is $(a^2 - 1)(b^2 - 1)/24$. This was proved by Tripathi in [12]. It is natural to wonder what would be the largest size of an (a, b, c)-core. Yang–Zhong–Zhou [17] found a formula for the largest size of an $(s, s+1, s+2)$-core. In Section 4, we give a formula for the largest size of a self-conjugate $(s, s + 1, s + 2)$-core partition. We also prove that such a partition is unique (see Theorem 3.3).

The layout of this paper is as follows. In Section 2, we introduce Johnson's c-coordinates and x-coordinates for core partitions. In Section 3, we give a formula for the largest size of a self-conjugate $(s, s + 1, s + 2)$ core partition. In Section 4, using c-coordinates, we count the number of a-core partitions with given largest part and second largest part. In Section 5, we derive formulas for the number of simultaneous core partitions by using Johnson's z-coordinates.

2. Review of Johnson's bijections

In this section, we review Johnson's bijections in [8], which are fundamental in this paper. For an integer a greater than 1, let $\mathbb{P}_a$ denote the set of a-core partitions. Let

$$C_a := \left\{(c_0, c_1, \dots, c_{a-1}) \in \mathbb{Z}^a : \sum_{i=0}^{a-1} c_i = 0 \right\}.$$

We first construct a bijective map from C_a to $\mathbb{P}_a$.