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List-edge-colouring planar graphs with precoloured edges

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ABSTRACT

Let G be a simple planar graph of maximum degree Δ, let t be a positive integer, and let L be an edge list assignment on G with $|L(e)| \geq \Delta + t$ for all $e \in E(G)$. We prove that if H is a subgraph of G that has been L-edge-coloured, then the edge-precolouring can be extended to an L-edge-colouring of G, provided that H has maximum degree $d \leq t$ and either $d \leq t - 4$ or Δ is large enough ($\Delta \geq 16 + d$ suffices). If $d > t$, there are examples for any choice of Δ where the extension is impossible.

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1. Introduction

In this paper all graphs are simple.

An *edge-colouring* of G is an assignment of colours to the edges of G so that adjacent edges receive different colours; if at most k colours are used we say it is a *k-edge-colouring*. The *chromatic index* of G, denoted $\chi'(G)$, is the minimum k such that G is k-edge-colourable. It is obvious that $\chi'(G) \geq \Delta$, where $\Delta := \Delta(G)$ is the maximum degree of G, and Vizing's Theorem [12] says that $\chi'(G) \leq \Delta + 1$.

In this paper we are looking to edge-colour a graph G, but with the constraint that some edges have already been coloured and cannot be changed. In this scenario we have no control over the edge-precolouring—if the edge-precoloured subgraph is H, then it will certainly have at least $\chi'(H)$ colours, but it could have many more, perhaps even more than $\chi'(G)$ colours. If we are looking to extend the edge-precolouring to a k-edge-colouring of G, then we will certainly need that k is at least the maximum degree of G, and that the edge-colouring of H uses at most k colours (i.e. is a k-edge-colouring). In general we consider the following question, first posed by Marcotte and Seymour [9]:

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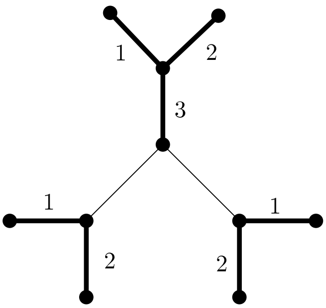

Fig. 1. A graph G with maximum degree $\Delta = 3$ with a precoloured subgraph of maximum degree Δ. In order to extend the edge-precolouring to a $(\Delta + t)$-edge-colouring of G we need $t \geq \Delta - 1$.

Question 1. *Given a graph G with maximum degree Δ and a subgraph H of G that has been $(\Delta+t)$-edge-coloured, can the edge-precolouring of H be extended to a $(\Delta + t)$-edge-colouring of G?*

Marcotte and Seymour's main result in [9] is a necessary condition for the answer to Question 1 to be "yes"; they prove that this condition is also sufficient when G is a multiforest (the condition is rather technical, so we do not state it here). Question 1 was shown to be NP-complete by Colbourn [4], and Marx [10] showed that this is true even when G is a planar 3-regular bipartite graph. Since, as Holyer [7] showed, it is NP-complete to decide whether $\chi'(G) = \Delta(G)$ or not, the special case $t = 0$ of Question 1 is also NP-complete for general graphs. In this paper we focus on Question 1 for planar graphs. Before saying more about planar graphs in particular however, let us make several quick observations about Question 1 in general.

Firstly, if t is huge – say at least $\Delta - 1$ – then the answer is *yes*, and moreover, the extension can be done greedily. This is because an edge in G sees at most $2(\Delta - 1)$ other edges, and when $t \geq \Delta - 1$, this value is at most $\Delta + t - 1$. If the maximum degree of H is Δ then this threshold for t is actually sharp. To see this, consider the graph G shown in Fig. 1, formed by taking a copy of $K_{1,\Delta}$ with one edge coloured Δ and the rest uncoloured, and joining each leaf to $\Delta - 1$ distinct new vertices via edges coloured $1, 2, \ldots, \Delta - 1$. Then G has maximum degree Δ, as does its edge-precoloured subgraph. However, in order to extend the edge-precolouring to a $(\Delta + t)$-edge-colouring of G, we need $\Delta - 1$ new colours, which forces $t \geq \Delta - 1$.

Given the above paragraph, Question 1 is only interesting when the maximum degree of H, say d, is strictly less than Δ. Here, we get a natural barrier to extension when $d > t$, via nearly the same example as above. Let G be the graph shown in Fig. 2, formed by taking a (uncoloured) copy of $K_{1,\Delta}$ and joining each leaf to $d < \Delta$ distinct new vertices, via edges coloured $1, 2, \ldots, d$. The resulting graph G has maximum degree Δ, and contains a precoloured subgraph H with maximum degree d. However, in order to extend the edge-precolouring to G, we need Δ new colours, meaning that for a $(\Delta + t)$-edge-colouring of G, we need $d \leq t$.

If it happened that H was edge-coloured efficiently (i.e. using at most $\chi'(H)$ colours), then our problem would be significantly reduced. In this special situation, one could use a completely new set of $\chi'(G - E(H))$ colours to extend to an edge-colouring of G with at most the following number of colours (according to Vizing's Theorem):

$$\chi'(G - E(H)) + \chi'(H) \leq \chi'(G) + \chi'(H) \leq \Delta + d + 2. \tag{1}$$

That is, when H has been edge-coloured efficiently, the answer to Question 1 is *yes* whenever $d \leq t-2$. Since extension can be impossible when $d > t$ (according to the above paragraph), this makes $d \in \{t - 1, t\}$ the only interesting values in this case, with further restrictions if any of the inequalities in (1) are strict. For example, if both G and H have chromatic index equal to their maximum degrees, then the colouring described above works whenever $d \leq t$, and hence we get a sharp threshold. Of course, this only works when H has been edge-precoloured efficiently, and in general we have no control over the edge-precolouring on H.

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