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On the critical densities of minor-closed classes

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ABSTRACT

Given a minor-closed class $\mathcal{A}$ of graphs, let $\beta_{\mathcal{A}}$ denote the supremum over all graphs in $\mathcal{A}$ of the ratio of edges to vertices. We investigate the set B of all such values $\beta_{\mathcal{A}}$, taking further the project begun by Eppstein. Amongst other results, we determine the small values in B (those up to 2); we show that B is 'asymptotically dense'; and we answer some questions posed by Eppstein.

1. Introduction

For a given graph G, let $v(G)$, $e(G)$ denote its number of vertices and edges, respectively. The *density* $\rho(G)$ of a graph G is $e(G)/v(G)$, the number of edges per vertex. Thus the average degree of G is $2\rho(G)$. Given a class $\mathcal{A}$ of graphs (closed under isomorphism), we let $\mathcal{A}_n$ denote the set of graphs in $\mathcal{A}$ on n vertices; let

$$e^*_{\mathcal{A}}(n) = \max_{G\in\mathcal{A}_n} e(G)$$

(where $e^*_{\mathcal{A}}(n) = 0$ if $\mathcal{A}_n$ is empty); and let

$$\beta_{\mathcal{A}} = \sup_{G\in\mathcal{A}} \rho(G) = \sup_{n\geqslant 1} e^*_{\mathcal{A}}(n)/n \quad \text{and} \quad \lambda_{\mathcal{A}} = \limsup_{n\to\infty} e^*_{\mathcal{A}}(n)/n.$$

A class of graphs is *proper* if it is non-empty and does not contain all graphs. A graph G contains a graph H as a *minor* if we can obtain a graph isomorphic to H from a subgraph of G by using edge contractions (discarding any loops and multiple edges, we are interested in simple graphs). A class $\mathcal{A}$ of graphs is *minor-closed* if whenever $G \in \mathcal{A}$ and H is a minor of G then H is in $\mathcal{A}$.

Let $\mathcal{A}$ be a proper minor-closed class of graphs. Then $\beta_{\mathcal{A}}$ is finite, as shown by Mader [11], and is called the *critical density* for $\mathcal{A}$. By Lemma 17 of [5], we always have $e^*_{\mathcal{A}}(n)/n \to \lambda_{\mathcal{A}}$ as $n \to \infty$:

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see also Norin [16], where limd($\mathcal{A}$) is the same as $2\lambda_{\mathcal{A}}$. Thus $\lambda_{\mathcal{A}}$ is called the *limiting density* of $\mathcal{A}$. By definition $\lambda_{\mathcal{A}} \leqslant \beta_{\mathcal{A}}$. Suppose for example that $\mathcal{A}$ is the minor-closed family of graphs G such that at most one component has a cycle, and any such component has at most 5 vertices: then $\lambda_{\mathcal{A}} = 1$ (because for $n \geqslant 5$ the densest graphs in $\mathcal{A}_n$ consist of K_5 and a tree on $n-5$ vertices), and $\beta_{\mathcal{A}} = 2$ (because K_5 is the densest graph in $\mathcal{A}$). If $\mathcal{A}$ is the class of series–parallel graphs (those with no minor K_4) then for each $n \geqslant 2$, each edge-maximal graph in $\mathcal{A}$ has $2n-3$ edges (see [14] for more on this), so $\lambda_{\mathcal{A}} = \beta_{\mathcal{A}} = 2$.

We are interested in the critical and limiting densities of proper minor-closed classes of graphs. The main object of study in this paper is the set of critical densities

$$B = \{\beta_{\mathcal{A}} : \mathcal{A} \text{ is a proper minor-closed class of graphs}\}.$$

We shall see shortly that for the corresponding set

$$L = \{\lambda_{\mathcal{A}} : \mathcal{A} \text{ is a proper minor-closed class of graphs}\}$$

of limiting densities we have $L = B$.

Given a proper minor-closed class $\mathcal{A}$ of graphs, a graph G is an *excluded minor* for $\mathcal{A}$ if G is not in $\mathcal{A}$ but each proper minor of G is in $\mathcal{A}$. If $\mathcal{H}$ is the set of excluded minors for $\mathcal{A}$, it is easy to see that $\mathcal{A}$ is the class of all graphs with no minor in $\mathcal{H}$: we write $\mathcal{A} = \mathrm{Ex}(\mathcal{H})$. By the Robertson–Seymour theorem [17], the set $\mathcal{H}$ is finite.

A class of graphs is called *decomposable* when a graph G is in the class if and only if each component of G is. It is easy to see that a minor-closed class of graphs is decomposable if and only if each excluded minor is connected. By Lemma 5 in [14], if $\mathcal{A}$ is a decomposable minor-closed class of graphs, then $e^*_{\mathcal{A}}(n)/n \to \beta_{\mathcal{A}}$ as $n \to \infty$, and so $\lambda_{\mathcal{A}} = \beta_{\mathcal{A}}$. Let

$$B_1 = \{\beta_{\mathcal{A}} : \mathcal{A} \text{ is a proper decomposable minor-closed class of graphs}\}$$

be the set of critical densities of decomposable minor-closed classes (so $B_1 \subseteq B$ trivially).

We call a graph G *minor-balanced* if each minor of G has density at most that of G, and *strictly minor-balanced* (or *density-minimal* [5]) if each proper minor has density strictly less than that of G. If G is minor-balanced with density $\rho(G) = \beta > 0$, and we let $\mathcal{A}$ be the decomposable class of graphs such that each component is a minor of G, then clearly $\beta_{\mathcal{A}} = \lambda_{\mathcal{A}} = \beta$. In this case, we say that the density β is *achievable*, and that β is the *maximum density* for $\mathcal{A}$. For example, K_5 is (strictly) minor-balanced with density 2, so 2 is an achievable density. Let

$$A = \{e(G)/v(G) : G \text{ is a minor-balanced graph}\}$$

be the set of densities of minor-balanced graphs.

The first theorem presented here is a general statement concerning the structure of the set B and describing the relationships between the sets A, B, B_1 and L. It is largely taken from Eppstein [5] and contains his Theorems 19 and 20. Given a set $S \subseteq \mathbb{R}$, we let $\bar{S}$ denote its closure and let S' denote the set of accumulation (or limit) points.

Theorem 1 (*a*). *The set B of critical densities is countable, closed and well-ordered by $<$; and (b) $\bar{A} = B = B_1 = L$ and $A' = B'$.*

Indeed it seems that more may be known, and each critical (or limiting) density is rational. This is given as Theorem 8.3 in Norin's survey [16], where a 'glimpse' of its lengthy proof is given, based on unpublished work of Kapadia and Norin: see also the recent paper of Kapadia [9] on minor-closed classes of matroids. For further results and conjectures see [16].

We shall see shortly that the set B is unbounded (indeed, by Theorem 5, if $\beta \in B$ then $1+\beta \in B$). It follows from the first part of Theorem 1 that B is nowhere dense. For, given $x \geqslant 0$, the set $\{\beta \in B : \beta > x\}$ has a least element x^+, and the non-empty open interval (x, x^+) is disjoint from B. In contrast, B is 'asymptotically dense', in the following sense. For each $x \geqslant 0$, let the 'gap above x' in B be $\delta_B(x) = x^+ - x$. Then B is asymptotically dense, in that $\delta_B(x) = o(1)$ as $x \to \infty$; and indeed we have the following result.