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Families of sets with no matchings of sizes 3 and 4

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ABSTRACT

In this paper, we study the following classical question of extremal set theory: what is the maximum size of a family of subsets of $[n]$ such that no s sets from the family are pairwise disjoint? This problem was first posed by Erdős and resolved for $n \equiv 0, -1 (\mathrm{mod}\ s)$ by Kleitman in the 60s. Very little progress was made on the problem until recently. The only result was a very lengthy resolution of the case $s = 3$, $n \equiv 1 \pmod 3$ by Quinn, which was written in his PhD thesis and never published in a refereed journal. In this paper, we give another, much shorter proof of Quinn's result, as well as resolve the case $s = 4$, $n \equiv 2 \pmod 4$. This complements the results in our recent paper, where, in particular, we answered the question in the case $n \equiv -2 \pmod s$ for $s \geq 5$.

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1. Introduction

Let $[n] := \{1, 2, \ldots, n\}$ be the standard n-element set and $2^{[n]}$ its power set. A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*. For $0 \leq k \leq n$, let $\binom{[n]}{k}$ denote the family of all k-subsets of $[n]$.

For a family $\mathcal{F}$, let $\nu(\mathcal{F})$ denote the maximum number of pairwise disjoint members of $\mathcal{F}$. Note that $\nu(\mathcal{F}) \leq n$ holds unless $\emptyset \in \mathcal{F}$. The fundamental parameter $\nu(\mathcal{F})$ is called the *independence number* or *matching number*.

Denote the size of the largest family $\mathcal{F} \subset 2^{[n]}$ with $\nu(\mathcal{F}) < s$ by $e(n, s)$. The following classical result was obtained by Kleitman.

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Kleitman's Theorem ([7]). Let $s \geq 2$, $m \geq 1$ be integers. Then the following holds.

$$\text{For } n = sm-1, \quad \text{we have} \quad e(n,s) = \sum_{m \leq t \leq n} \binom{n}{t}, \tag{1}$$

$$\text{for } n = sm, \quad \text{we have} \quad e(n,s) = \frac{s-1}{s}\binom{n}{m} + \sum_{m+1 \leq t \leq n} \binom{n}{t}. \tag{2}$$

The value $e(ms-1, s)$ is attained on the family of all sets of size greater than or equal to m. The following matching example for (2) was proposed by Kleitman:

$$\big\{K \subset [sm] : |K| \geq m+1\big\} \cup \binom{[sm-1]}{m}.$$

(Note that $\binom{sm-1}{m} = \frac{s-1}{s}\binom{sm}{m}$.) Let us mention that for $s = 2$ both bounds (1) and (2) reduce to $e(n,2) = 2^{n-1}$. This easy statement was proved already by Erdős, Ko and Rado [1].

Although (1) and (2) are beautiful results, for $s \geq 3$ they leave open the cases of $n \not\equiv 0, -1 (\mathrm{mod}\, s)$. For $s = 3$, the only remaining case was solved by Quinn [8]. However, his argument is very lengthy and was never published in a refereed journal. In this paper, we reprove his result, as well as extend it to the case $n = 4m+2$, $s = 4$.

Theorem 1. *Fix an integer $m \geq 1$. Then for $s = 3, 4$ and $n = sm+s-2$ we have*

$$e(n,s) = \binom{n-1}{m-1} + \sum_{m+1 \leq t \leq n} \binom{n}{t}. \tag{3}$$

The following s-matching-free family shows that "$\geq$" holds in the equality above for any $s \geq 3$ and $n = sm+s-2$.

$$\big\{L \subset [n] : |L| \geq m+1\big\} \cup \Big\{L \in \binom{[n]}{m} : 1 \in L\Big\}.$$

Theorem 1 bridges the gap that was left between Quinn's result and the result of the paper [2], where we verified the same statement for $n = sm+s-2, s \geq 5$. Contrary to the intuition, the problem gets easier as s becomes larger, and thus the proof for $s = 3, 4$ is more intricate than that of [2].

The proof is based on a non-trivial averaging technique somewhat in the spirit of Katona's circle method [6]: we choose a certain configuration of sets, show that the intersection of a family satisfying the conditions of Theorem 1 with *each* such configuration cannot be too large and then average over all such configurations. However, the configuration is quite complicated, the sets in the configuration actually have weights, and, in order to bound the weighted intersection of the family with each configuration, we use some kind of discharging method.

The method we develop here has proved to be very useful and was already used in several papers. In a recent paper [4], we applied it to completely resolve the following problem studied by Kleitman: what is the maximum cardinality of a family $\mathcal{F} \subset 2^{[n]}$ that does not contain two disjoint sets F_1, F_2, along with their union $F_1 \cup F_2$? We refer the reader to the papers [2,4] for a more detailed introduction to the topic and, in particular, to [2] for the discussion of the value of $e(n,s)$ for general n, s. See also [3], where the method we developed was applied.

We note that (1) and (2), along with more general statements, are proved using a simpler version of our technique in [5].

2. Preliminaries

Recall that $\mathcal{F}$ is called an *up-set* if for any $F \in \mathcal{F}$ all sets that contain F are also in $\mathcal{F}$. Since we aim to upper bound the sizes of families $\mathcal{F}$ with $\nu(\mathcal{F}) < s$, we may restrict our attention to the families that are up-sets, which we assume for the rest of the paper.

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