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Explicit computation of some families of Hurwitz numbers

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ABSTRACT

We compute the number of (weak) equivalence classes of branched covers from a surface of genus g to the sphere, with 3 branching points, degree $2k$, and local degrees over the branching points of the form $(2, \ldots, 2)$, $(2h+1, 1, 2, \ldots, 2)$, $\pi = (d_i)_{i=1}^{\ell}$, for several values of g and h. We obtain explicit formulae of arithmetic nature in terms of the local degrees d_i. Our proofs employ a combinatorial method based on Grothendieck's *dessins d'enfant*.

In this introduction we describe the enumeration problem faced in the present paper and the situations in which we solve it.

Surface branched covers. A surface branched cover is a map

$$f : \widetilde{\Sigma} \to \Sigma$$

where $\widetilde{\Sigma}$ and Σ are closed and connected surfaces and f is locally modeled on maps of the form

$$(\mathbb{C}, 0) \ni z \mapsto z^m \in (\mathbb{C}, 0).$$

If $m > 1$ the point 0 in the target $\mathbb{C}$ is called a *branching point*, and m is called the local degree at the point 0 in the source $\mathbb{C}$. There are finitely many branching points, removing which, together with their pre-images, one gets a genuine cover of some degree d. If there are n branching points, the local degrees at the points in the pre-image of the jth one form a partition π_j of d of some length ℓ_j, and the following Riemann-Hurwitz relation holds:

$$\chi\left(\widetilde{\Sigma}\right) - (\ell_1 + \cdots + \ell_n) = d\left(\chi\left(\Sigma\right) - n\right).$$

Let us now call *branch datum* a 5-tuple

$$\left(\widetilde{\Sigma}, \Sigma, d, n, \pi_1, \ldots, \pi_n\right)$$

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Table 1
Values of g, h, ℓ giving compatible data (♡).

	$g=0$	$g=1$	$g=2$	$g=3$
$2h+1=1$	$\ell=\mathbf{1}$	–	–	–
$2h+1=3$	$\ell=\mathbf{2}$	–	–	–
$2h+1=5$	$\ell=\mathbf{3}$	$\ell=\mathbf{1}$	–	–
$2h+1=7$	$(\ell=4)$	$\ell=\mathbf{2}$	–	–
$2h+1=9$	$(\ell=5)$	$(\ell=3)$	$\ell=\mathbf{1}$	–
$2h+1=11$	$(\ell=6)$	$(\ell=4)$	$(\ell=2)$	–
$2h+1=13$	$(\ell=7)$	$(\ell=5)$	$(\ell=3)$	$(\ell=1)$

and let us say it is *compatible* if it satisfies the Riemann-Hurwitz relation. (For a non-orientable $\widetilde{\Sigma}$ and/or Σ this relation should actually be complemented with certain other necessary conditions, but we restrict to an orientable Σ in this paper, so we do not spell out these conditions here.)

The Hurwitz problem. The very old *Hurwitz problem* asks which compatible branch data are *realizable* (namely, associated to some existing surface branched cover) and which are *exceptional* (non-realizable). Several partial solutions to this problem have been obtained over the time, and we quickly mention here the fundamental [3], the survey [16], and the more recent [13–15,2,17]. In particular, for an orientable Σ the problem has been shown to have a positive solution whenever Σ has positive genus. When Σ is the sphere S, many realizability and exceptionality results have been obtained (some of experimental nature), but the general pattern of what data are realizable remains elusive. One guiding conjecture in this context is that *a compatible branch datum is always realizable if its degree is a prime number*. It was actually shown in [3] that proving this conjecture in the special case of 3 branching points would imply the general case. This is why many efforts have been devoted in recent years to investigating the realizability of compatible branch data with base surface Σ the sphere S and having $n=3$ branching points. See in particular [14,15] for some evidence supporting the conjecture.

Hurwitz numbers. Two branched covers

$$f_1:\widetilde{\Sigma}\to\Sigma \qquad f_2:\widetilde{\Sigma}\to\Sigma$$

are said to be *weakly equivalent* if there exist homeomorphisms $\widetilde{g}:\widetilde{\Sigma}\to\widetilde{\Sigma}$ and $g:\Sigma\to\Sigma$ such that $f_1\circ\widetilde{g}=g\circ f_2$, and *strongly equivalent* if the set of branching points in Σ is fixed once and forever and one can take $g=\mathrm{id}_\Sigma$. The *(weak or strong) Hurwitz number* of a compatible branch datum is the number of (weak or strong) equivalence classes of branched covers realizing it. So the Hurwitz problem can be rephrased as the question whether a Hurwitz number is positive or not (a weak Hurwitz number can be smaller than the corresponding strong one, but they can only vanish simultaneously). Long ago Mednykh in [10,11] gave some formulae for the computation of the strong Hurwitz numbers, but the actual implementation of these formulae is rather elaborate in general. Several results were also obtained in more recent years in [4,7,6,8,12].

Computations. In this paper we consider branch data of the form

$$(\heartsuit)\ \left(\widetilde{\Sigma},\ \Sigma=S,\ d=2k,\ n=3,\ (2,\ldots,2),\ (2h+1,1,2,\ldots,2),\ \pi=(d_i)_{i=1}^{\ell}\right)$$

for $h\geqslant 0$. A direct computation shows that such a datum is compatible for $h\geqslant 2g$, where g is the genus of $\widetilde{\Sigma}$, and $\ell=h-2g+1$. We compute the weak Hurwitz number of the datum for the values of $g, 2h+1, \ell$ shown in boldface in Table 1. More values could be obtained, including for instance those within parentheses in the table, using the same techniques as we employ below, but the complication of the topological and combinatorial situation grows very rapidly, and the arithmetic formulae giving the weak Hurwitz numbers are likely to be rather intricate for larger values of g and h.

For brevity we will henceforth denote by ν the number of weakly inequivalent realizations of (♡) for any given values of g and h.

Theorem 0.1. *For $g=0$ and $0\leqslant h\leqslant 2$ there hold:*

- *For $h=0$, whence $\ell=1$ and $\pi=(2k)$, we always have $\nu=1$;*
- *For $h=1$, whence $\ell=2$ and $\pi=(p,2k-p)$ with $p\leqslant k$, we have $\nu=1$ for $p<k$, and $\nu=0$ for $p=k$;*