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A bound on the inducibility of cycles

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ABSTRACT

In 1975, Pippenger and Golumbic conjectured that every n-vertex graph has at most $n^k/(k^k - k)$ induced cycles of length $k \geq 5$. We prove that every n-vertex graph has at most $2n^k/k^k$ induced cycles of length k.

1. Introduction

The study of the number of induced copies of a given graph is a classical topic in extremal combinatorics, which can be traced back to the work of Pippenger and Golumbic [9] from 1975. The *induced density* of a graph H in a graph G, which is denoted by $i(H, G)$, is the number of induced copies of H in G divided by $\binom{|V(G)|}{|V(H)|}$.

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A standard averaging argument shows that for all graphs H and G and all integers $|V(H)| \leq n < |V(G)|$, there exists an n-vertex graph G' such that $i(H, G') \geq i(H, G)$. It follows that the sequence $i(H, n)$ is monotone non-increasing in n, and hence it converges for every H. The *inducibility* of a graph H, which is denoted by $\mathrm{ind}(H)$, is the limit of the sequence $i(H, n)$ where $i(H, n)$ is the maximum induced density of H in an n-vertex graph.

Pippenger and Golumbic [9] showed that the inducibility of every k-vertex graph H is at least $k!/(k^k - k)$ and conjectured that this bound is tight for a cycle of length $k \geq 5$.

Conjecture 1 *(Pippenger and Golumbic [9]). The inducibility of a cycle C_k of length $k \geq 5$ is equal to $\frac{k!}{k^k - k}$.*

In the recent years, the flag algebra method of Razborov [10] led to new bounds on the inducibility of small graphs [1,7], which included the proof of Conjecture 1 for $k = 5$ by Balogh et al. [1]. Other classes of graphs for which the inducibility has been determined include sufficiently balanced complete multipartite graphs [2–4,9] and sufficiently large balanced blow-ups of arbitrary graphs [5].

Motivated by Conjecture 1, we study the inducibility of cycles and provide a new upper bound. In their original paper, Pippenger and Golumbic [9] proved Conjecture 1 within a multiplicative factor of $2e$, i.e., they proved that

$$\mathrm{ind}(C_k) \leq \frac{2k!}{k(k-1)^{k-1}} = (2e + o(1))\,\frac{k!}{k^k}.$$

The multiplicative factor $2e$ has recently been improved to $128e/81$ by Hefetz and Tyomkyn [6] and to e by Pfender and Phillips [8]. Our main result reads as follows.

Theorem 1. *Every n-vertex graph G contains at most $2n^k/k^k$ induced copies of a cycle C_k of length $k \geq 5$.*

This attains the bound of Conjecture 1 up to a multiplicative factor of 2, i.e., we show that

$$\mathrm{ind}(C_k) \leq (2 + o(1))\,\frac{k!}{k^k}. \tag{1}$$

We remark that we convinced ourselves that more detailed arguments could be used to improve the multiplicative factor 2 in (1) to $2 - \varepsilon$ for some tiny $\varepsilon > 0$ but we do not include further details to keep this note short and easily accessible.

2. Proof of Theorem 1

The rest of the paper is devoted to the proof of Theorem 1. Fix an n-vertex graph G and an integer $k \geq 5$. Instead of counting the number of induced copies of C_k, we will