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Discrete groups without finite quotients

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ABSTRACT

We construct an infinite discrete subgroup of the isometry group of $\mathbb{H}^3$ with no finite quotients other than the trivial group.

It is well-known that every finitely generated linear group is residually finite [9]. Finite generation is definitively necessary, as it is already made apparent by the group $\mathbb{Q}$. However, people working with Kleinian groups, that is with discrete groups of isometries of hyperbolic spaces $\mathbb{H}^n$, might find examples as $\mathbb{Q}$ to be kind of pathological. In fact, it is well-known that discreteness of a group of isometries of hyperbolic space imposes non-trivial algebraic conditions. For example, centralisers of infinite order elements in discrete Kleinian groups are virtually abelian. Or, more to the point, while $\mathrm{PSL}_2\,\mathbb{Q} \subset \mathrm{PSL}_2\,\mathbb{R} \subset \mathrm{Isom}(\mathbb{H}^2)$ is simple [5,7], it is easy to see, using small cancellation arguments, that there are no infinite, simple, and discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ (compare with [3,4]). Also, Kleinian groups are mostly studied in low dimensions, and in that setting further algebraic restrictions do arise. For instance, arbitrary discrete subgroups of $\mathrm{Isom}\,\mathbb{H}^2$ are residually finite.

The goal of this note is to present examples of discrete subgroups of $\mathrm{Isom}\,\mathbb{H}^3$ which fail to be residually finite. In fact, they don't have any non-trivial finite quotients whatsoever.

Example 1. There is an infinite discrete subgroup $G \subset \mathrm{Isom}(\mathbb{H}^3)$ without finite non-trivial quotients.

As we just said, having no finite quotients, the group G in Example 1 clearly fails to be residually finite. Examples of discrete, non-residually finite subgroups of $\mathrm{Isom}(\mathbb{H}^3)$ have been previously constructed by Agol [1]. Both Agol's examples and the group in Example 1 have torsion. We present next an example,

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a variation of Agol's example, showing that there are also torsion free discrete non-residually finite subgroups of $\mathrm{Isom}(\mathbb{H}^3)$:

Example 2. There is a torsion free discrete subgroup $G \subset \mathrm{Isom}(\mathbb{H}^3)$ which is not residually finite.

The remaining of this note is devoted to discuss these two examples.

Discussion of the examples

In the course of our discussion we feel free to use standard facts of hyperbolic geometry as one might find in classical texts such as [6,8]. It will also be convenient to see our groups as fundamental groups of infinite, locally finite, graph of groups. We refer to standard texts like [10] for basic facts about graphs of groups.

Example 1

We give an algebraic description of a group G, then we prove that it has no finite quotients, and we finally show that it is isomorphic to a discrete subgroup of $\mathrm{PSL}_2\,\mathbb{C}$.

Let T be the maximal rooted binary tree. Denote by $\mathcal{V}$ and $\mathcal{E}$ the sets of vertices and edges respectively, let $*$ be the root of T and, for $v \in \mathcal{V}$, let $|v| \in \mathbb{N}$ be the distance from v to $*$. We orient the edges of T so that they point to the root and for $e \in \mathcal{E}$ we let e^+ be its terminal vertex. Given a vertex $v \in \mathcal{V}$ with $|v| \geq 1$ let $e_0(v)$ the edge leaving v and pointing out of v and label the two edges pointing into v by $e_1(v)$ and $e_2(v)$.

Consider from now the group

$$G = \left\langle \{g_e | e \in \mathcal{E}\} \middle| \left\{ g_{e_0(v)}^{3+|v|},\ g_{e_0(v)} g_{e_1(v)}^{-1} g_{e_2(v)}^{-1} \middle| v \in \mathcal{V} \text{ with } |v| \geq 1 \right\} \right\rangle.$$

The group G also admits a description as the fundamental group

$$G = \pi_1(\mathcal{T})$$

of a graph of groups $\mathcal{T}$ with underlaying graph T, with vertex groups

$$G_v = \left\langle g_{e_0(v)}, g_{e_1(v)}, g_{e_2(v)} \middle| g_{e_0(v)}^{3+|v|}, g_{e_1(v)}^{4+|v|}, g_{e_2(v)}^{4+|v|}, g_{e_0(v)} g_{e_1(v)}^{-1} g_{e_2(v)}^{-1} \right\rangle$$

if $v \neq *$, with

$$G_* = \left\langle g_{e_1(*)}, g_{e_2(*)} \middle| g_{e_1(*)}^4, g_{e_2(*)}^4 \right\rangle,$$

and with edge groups

$$G_e = \left\langle g_e \middle| g_e^{4+|e^+|} \right\rangle.$$

We are going to think of the group G as the nested union of a sequence of subgroups. The easiest way to describe these subgroups is as the fundamental groups

$$G^n = \pi_1(\mathcal{T}^n)$$

of the subgraph of groups $\mathcal{T}^n \subset \mathcal{T}$ corresponding to the ball of radius n around the root $*$. Alternatively, G^n is the subgroup of G generated by all those elements $g_{e_0(v)}$ with $|v| \leq n$. We have

$$G^0 \subset G^1 \subset G^2 \subset G^3 \subset \ldots, \quad G = \bigcup_{n\in\mathbb{N}} G^n.$$