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Conservative or dissipative quasi-interpolation method for evolutionary partial differential equations☆

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ABSTRACT

Based on quasi-interpolation, we propose a new meshless conservative or dissipative method for nonlinear time-dependent partial differential equations. Using the method of lines, we first discretize the equation in space with the quasi-interpolation method, then employ the average vector field method in time discretization to derive the final numerical scheme. The method not only inherits the conservation or dissipation property of the equation but also has the meshless feature since we use the nonuniform grids in spatial discretization. Several numerical examples are presented to demonstrate the effectiveness of the proposed method.

1. Introduction

In this paper, we are interested in the numerical approximation to the evolutionary partial differential equations of the following form:

$$\frac{\partial u}{\partial t} = \mathcal{D}\frac{\delta G}{\delta u}. \tag{1.1}$$

Here $u = u(x,t)$, $(x,t) \in \mathbb{R}^d \times \mathbb{R}$, $\mathcal{D}$ is a skew-symmetric or negative semidefinite differential operator, G is a functional defined by

$$G = \int_{\mathbb{R}^d} g(x; u, u_x, u_{xx}, \ldots)dx,$$

and $\delta G/\delta u$ denotes the variational derivative of G.

In the past few decades, the systematic methods for designing numerical scheme that preserve energy or dissipation property have been intensively studied. Furihata [7] proposed a finite difference method for sloving the system $\frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial x}\right)^{\alpha} \frac{\delta G}{\delta u}$ which inherits the energy conservation or dissipation property. Celledoni et al. [5] developed a more general framework for obtaining the conservative or dissipative discretization of (1.1) and illustrated the method by using the finite difference method and spectral method. Furthermore, Matsuo [19] established an adaptive conservative or dissipative numerical method based on Galerkin method for nonlinear evolutionary partial differential equations.

However, all the methods presented above have some limitations when applied to nonuniform grids. Hence, we devote to proposing a simple and effective discretization method performing on nonuniform grids. The quasi-interpolation method is a good candidate which has been widely studied in the literature to process scattered centers. It was first proposed by Hardy [15] for the design of aircraft and then developed by many authors, Beatson and Powell [1], Beatson and Dyn [2], Buhmann [3,4], Wu and Schaback [23], just name a few. Moreover, it has been applied broadly to various fields, such as numerical solution for differential equations [6,9,16], boundary detection [8], symplectic scheme [24] and so on.

The most salient feature of quasi-interpolation method is that it provides the approximation solution directly without the requirement to solve the linear system of equation. Besides, it approximates high-order derivatives in a stable way compared with the finite difference method [17,18]. And it is extremely suitable for nonuniform grids. Hence, we employ the quasi-interpolation method in the spatial dicretization of the Eq. (1.1).

The procedure of our method can be described in the following steps. Firstly, we discretize the energy functional based on quasi-interpolation method. Then we derive the semi-discretized system from this discrete energy functional. Finally, employing average vector field method in temporal domain to obtain the full-discretized scheme.

The paper is organized as follows. Section 2 provides the basic theory of the evolutionary partial differential equations and quasi-interpolation. The conservative or dissipative discretization method is discussed in Section 3. Several examples are given in Section 4 to illustrate the performance of the method. Section 5 concludes the paper.

2. Preliminaries

2.1. Evolutionary partial differential equations

Consider the evolutionary partial differential equations of the form (1.1) with $\mathcal{D}$ a skew-symmetric or negative semidefinite operator. If $\mathcal{D}$

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is a skew-symmetric operator, then the energy of the system is invariant with respect to time, namely

$$\frac{dG}{dt} = \left\langle \frac{\delta G}{\delta u}, \frac{\partial u}{\partial t} \right\rangle = \left\langle \frac{\delta G}{\delta u}, \mathcal{D}\frac{\delta G}{\delta u} \right\rangle = 0.$$

The typical examples for this kind of equations are Hamiltonian wave equation, Schrödinger equation and Maxwell equation. In this case, we use the operator $\mathcal{S}$ to replace $\mathcal{D}$.

If $\mathcal{D}$ is negative semidefinite, then the system has a dissipation property

$$\frac{dG}{dt} = \left\langle \frac{\delta G}{\delta u}, \frac{\partial u}{\partial t} \right\rangle = \left\langle \frac{\delta G}{\delta u}, \mathcal{D}\frac{\delta G}{\delta u} \right\rangle \le 0.$$

Many famous equations including heat equation, Allen–Cahn equation, Cahn–Hilliard equation, Ginzburg–Landau equation can de derived from the general form (1.1) by taking different operators $\mathcal{D}$ and functionals G. In this case, we denote $\mathcal{D}$ by $\mathcal{N}$.

As we can see, under appropriate boundary conditions, the energy functional of the system is conservative or dissipative with respect to time. So one might expect that the numerical scheme also has the energy conservation or dissipation property to provide more accurate solution satisfying the physical law. In this paper, we will use quasi-interpolation method in space discretization to derive conservative or dissipative numerical schemes.

2.2. Quasi-interpolation

In this section, we briefly introduce the quasi-interpolation method. For more details, one can refer to [3,10–14,22] and the references therein.

Given sampling data $\{x_j, f(x_j)\}$, the quasi-interpolant can be constructed as [22]

$$f^*(x) = \sum_j f(x_j)\psi(x - x_j)\Delta_j, \tag{2.1}$$

where $\psi(x)$ is a symmetric kernel with certain smoothness and decaying conditions and satisfies

$$\int_{\mathbb{R}^d} \psi(x)dx = 1,$$

$\{\Delta_j\}$ are positive weights of quadrature satisfy

$$\int_{\mathbb{R}^d} f(x)dx \approx \sum_j f(x_j)\Delta_j.$$

The quasi-interpolation method gives the approximation solution directly without solving linear system. So it is very simple and consumes less time. Moreover, the approximation order of the quasi-interpolation $f^*(x)$ to $f(x)$ was analyzed in [22]. Furthermore, [17] gave the convergence order of the kth derivatives $(f^*)^{(k)}(x)$ to $f^{(k)}(x)$. They also verified that the quasi-interpolation method is more stable than finite difference method.

3. Conservative or dissipative discretization method

In this section, we present the conservative or dissipative discretization method of the general evolutionary PDEs of the form (1.1). We first discretize the energy functional by using quasi-interpolation and then obtain the semi-discretized equations. Finally, we employ the average vector field (AVF) method for temporal discretization.

3.1. Space discretization using quasi-interpolation method

In spatial domain, we firstly discretize the integral

$$G = \int_{\mathbb{R}^d} g(x; u, u_x, u_{xx}, \dots)dx$$

by using quasi-interpolation method.

Given data $\{u(x_j, t)\}$, we construct the quasi-interpolation

$$u^*(x,t) = \sum_j u(x_j,t)\psi(x - x_j)\Delta_j. \tag{3.1}$$

Then one can obtain the approximation of mth derivative

$$u^{(m)}(x_k,t) \approx (u^*)^{(m)}(x_k,t) = \sum_j u(x_j,t)\psi^{(m)}(x_k - x_j)\Delta_j. \tag{3.2}$$

Next we approximate the L^2 inner product by quadrature to arrive at a weighted inner product

$$\langle u, v\rangle_{L^2} = \int u(x)v(x)dx \approx \sum_{j=1}^{N} u(x_j)v(x_j)\Delta_j = U^T\Delta V = \langle U, V\rangle_\Delta,$$

where $U = (\dots, u(x_j,t), \dots)^T$ and $\Delta = \mathrm{diag}(\Delta_j)$.

Finally, we can obtain the discretization of integral G which can be characterized as the following form

$$G_p = G_p(U; \Delta, \Psi_1, \dots, \Psi_m), \tag{3.3}$$

where $\Psi_m = (\psi^{(m)}(x_k - x_j))$.

In order to derive the spatial discretization of the equation, we need to characterize the discrete variational derivative of G_p. According to the definition of the discrete variational derivative

$$\left\langle \frac{\delta G_p}{\delta U}, V \right\rangle_\Delta = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} G_p(U + \epsilon V),\ \forall V \in \mathbb{R}^N,$$

which means that

$$\left(\frac{\delta G_p}{\delta U}\right)^T \Delta V = (\nabla G_p(U))^T V,\ \forall V \in \mathbb{R}^N,$$

we conclude that

$$\frac{\delta G_p}{\delta U} = \Delta^{-1}\nabla G_p(U).$$

In the followings, we discuss the conservative case and the dissipative case separately.

Conservative type. The equation is defined as

$$\frac{\partial u}{\partial t} = \mathcal{S}\frac{\delta G}{\delta u}. \tag{3.4}$$

Using $\frac{\delta G_d}{\delta U}$ as an approximation to $\frac{\delta G}{\delta u}$ and approximating $\mathcal{S}$ by matrix $\mathcal{S}_p$ which is skew-symmetric with respect to $\langle\cdot, \cdot\rangle_\Delta$, we have the following semi-discretized system

$$\frac{dU}{dt} = \mathcal{S}_p\Delta^{-1}\nabla G_p(U). \tag{3.5}$$

The energy conservation can be described in the following theorem.

Theorem 3.1. *If U satisfies the system (3.5) and G_p is defined as (3.3), then the following equality holds*

$$\frac{dG_p}{dt} = 0.$$

Proof. With the definition of G_p, we have

$$\frac{dG_p}{dt} = (\nabla G_p(U))^T U_t.$$

Using the following relation $U_t = \mathcal{S}_p\Delta^{-1}\nabla G_p(U)$, we can conclude that

$$\frac{dG_p}{dt} = (\nabla G_p(U))^T \mathcal{S}_p\Delta^{-1}\nabla G_p(U).$$

Since $\mathcal{S}_p$ is skew-symmetric with respect to inner product $\langle\cdot, \cdot\rangle_\Delta$, we have

$$(\mathcal{S}_p\Delta^{-1})^T = \Delta^{-1}(\mathcal{S}_p)^T\Delta\Delta^{-1} = -\Delta^{-1}\Delta\mathcal{S}_p\Delta^{-1} = -\mathcal{S}_p\Delta^{-1}.$$

This shows that the matrix $\mathcal{S}_p\Delta^{-1}$ is also skew-symmetric. Hence the theorem holds. □

Dissipative type. The equation is defined as

$$\frac{\partial u}{\partial t} = \mathcal{N}\frac{\delta G}{\delta u}. \tag{3.6}$$