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Unified convergence analysis of frozen Newton-like methods under generalized conditions

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ABSTRACT

The objective in this article is to present a unified convergence analysis of frozen Newton-like methods under generalized Lipschitz-type conditions for Banach space valued operators. We also use our new idea of restricted convergence domains, where we find a more precise location, where the iterates lie leading to at least as tight majorizing functions. Consequently, the new convergence criteria are weaker than in earlier works resulting to the expansion of the applicability of these methods. The conditions do not necessarily imply the differentiability of the operator involved. This way our method is suitable for solving equations and systems of equations. Numerical examples complete the presentation of this article.

1. Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $\mathcal{D}$ be a subset of $\mathcal{X}$. We denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of all bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. Let $F : \mathcal{D} \subset \mathcal{X} \longrightarrow \mathcal{Y}$ be a continuous nonlinear operator. Consider the equation

$$F(x) = 0. \tag{1.1}$$

The task of obtaining a locally unique solution p of equation $F(x) = 0$ is very important. Indeed, using mathematical modeling [1–3] numerous problems in optimization, control theory, inverse theory, Mathematical physics, Chemistry, Biology, Economics and also in Engineering, can be made to look like equation $F(x) = 0$ defined on suitable abstract spaces. It is desirable to find p in closed form. However, this task can be achieved only in special cases. That explains why most researchers resort to iterative methods which generate a sequence converging to p under certain conditions.

It is well known [2] that under suitable conditions, Newton's method defined for each $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \tag{1.2}$$

where $x_0 \in \mathcal{D}$ is an initial point provides a quadratically convergent iteration $\{x_n\}$ for solving Eq. (1.1). But, there is a plethora of problems that for some reasons, Newton's method cannot apply in its original form. A case of interest occurs when the derivative is not continuously invertible, as for instance, when dealing with small divisors [4]. That is why numerous

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authors have proposed variants of Newton's method which converge under Lipschitz-type conditions provided that certain Kantorovich-type criteria are satisfied.

In this article, we study the unifying class of frozen Newton-like methods defined for each $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - L(y_{t_n})^{-1}F(x_n), \tag{1.3}$$

where x_0 is an initial point, $L(.) : \mathcal{D} \longrightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})$, t_n is a nondecreasing sequence of integers satisfying the conditions

$$t_0 = 0,\ \ t_n \leq n \text{ for each } n = 0, 1, 2, \ldots \tag{1.4}$$

and y_{t_n} is the highest indexed point $x_0, x_1, \ldots, x_{t_n}$ for which $L(y_{t_n})^{-1}$ exists. Suppose that $L(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. It is well known that from the numerical efficiency point of view, it is not advantageous to change the operator $L(y_{t_n})^{-1}$ at each step of the iterative method. We obtain more efficient iterative methods, if we keep this operator piece wise constant. According to the dimension of the space optimal methods can be obtained [3]. Many popular methods can be obtained from method (1.3) with an appropriate choice of the sequence $\{t_n\}$ for each $n = 0, 1, 2 \ldots$:

SINGLE POINT METHODS:
Newton's method (1.2):

$$t_n = n,\ L(y_{t_n}) = F'(x_n).$$

Modified Newton's method:

$$\begin{aligned} x_{n+1} &= x_n - F'(x_0)^{-1}F(x_n),\\ t_n = 0, \qquad L(y_{t_n}) &= F'(x_0). \end{aligned}$$

Stirling's method for the equation $G(x) = x$, $\mathcal{X} = \mathcal{Y}$ and $F(x) = x - G(x)$:

$$\begin{aligned} x_{n+1} &= x_n - F'(x_n)^{-1}F(x_n),\\ t_n = n, \qquad L(y_{t_n}) &= F'(x_n). \end{aligned}$$

Picard's method [5]:

$$\begin{aligned} x_{n+1} &= x_n - F(x_n),\\ t_n = n, \qquad L(y_{t_n}) &= I, \end{aligned}$$

where I is the identity operator on $\mathcal{X}$ and $\mathcal{X} = \mathcal{Y}$.

Traub method: $t_{km+j} = km$, $j = 0, 1, \ldots, m-1$, $k = 0, 1, 2, \ldots$. Then, method (1.3) reduces to an iterative method studied by Traub [3]. The parameter m is chosen according to the dimension of the space in order to maximize the numerical efficiency of the method [6].

TWO POINT METHODS:
We can write method (1.3) in the form

$$x_{n+1} = x_n - L(y_{s_n}, y_{t_n})^{-1}F(x_n), \tag{1.5}$$

where $L(.) = L(y_{s_n}, .) : \mathcal{D} \longrightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y})$, s_n is a nondecreasing sequence of integers satisfying the conditions

$$s_0 = -1,\ s_n \leq t_n \leq n \text{ for each } n = 1, 2, \ldots. \tag{1.6}$$

Secant method:

$$\begin{aligned} x_{n+1} &= x_n - [x_{n-1}, x_n; F]^{-1}F(x_n),\\ s_n &= n-1,\ t_n = n,\ [.,.; F] : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y}) \end{aligned}$$

is a consistent approximation of the Fréchet derivative F' of F and $L(y_{s_n}, y_{t_n}) = [x_{n-1}, x_n; F]$.

Modified secant method:

$$\begin{aligned} x_{n+1} &= x_n - [x_{-1}, x_0; F]^{-1}F(x_n),\\ s_n &= -1,\ t_n = 0,\ \text{for each } n = 0, 1, 2, \ldots \end{aligned}$$

and $L(y_{s_n}, y_{t_n}) = [x_{-1}, x_0; F]$.

Traub method [3]:
$s_{km+j} = km - 1$, $t_{km+j} = km$, $s_{-1} = s_0 = -1$, $j = 0, 1, 2, \ldots, m-1$, $k = 0, 1, 2, \ldots$. Then, method (1.5) reduces to a procedure considered by Traub for scalar equations [3].