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Moving mesh simulation of contact sets in two dimensional models of elastic–electrostatic deflection problems

Kelsey L. DiPietro [a], Ronald D. Haynes [b], Weizhang Huang [c], Alan E. Lindsay [a,*], Yufei Yu [c]

[a] Department of Applied and Computational Mathematics & Statistics, University of Notre Dame, Notre Dame, IN 46656, USA
[b] Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL A1C 5S7, Canada
[c] Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA

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ABSTRACT

Numerical and analytical methods are developed for the investigation of contact sets in electrostatic–elastic deflections modeling micro-electro mechanical systems. The model for the membrane deflection is a fourth-order semi-linear partial differential equation and the contact events occur in this system as finite time singularities. Primary research interest is in the dependence of the contact set on model parameters and the geometry of the domain. An adaptive numerical strategy is developed based on a moving mesh partial differential equation to dynamically relocate a fixed number of mesh points to increase density where the solution has fine scale detail, particularly in the vicinity of forming singularities. To complement this computational tool, a singular perturbation analysis is used to develop a geometric theory for predicting the possible contact sets. The validity of these two approaches are demonstrated with a variety of test cases.

1. Introduction

The present work is concerned with numerical simulation and singular perturbation analysis of the initial-boundary value problem of a fourth-order parabolic partial differential equation (PDE)

$$\begin{cases} u_t = -\varepsilon^2 \Delta^2 u - \dfrac{1}{(1+u)^2}, & (\mathbf{x},t) \in \Omega \times (0,T); \\ u = \Delta u = 0, & (\mathbf{x},t) \in \partial\Omega \times (0,T); \\ u(\mathbf{x},0) = 0, & \mathbf{x} \in \Omega; \end{cases} \tag{1}$$

in a variety of bounded two dimensional geometries Ω. The system (1) models the non-dimensional vertical deflection $z = u(\mathbf{x})$ for $\mathbf{x} = (x, y) \in \Omega$ of a Micro-electro mechanical systems (MEMS) capacitor [30,35,37]. The MEMS capacitor is a key component of modern nanotechnology [3,44,45] that features a deformable elastic membrane held fixed above a rigid substrate (see Fig. 1(a)). When an electric potential is applied between the deflecting plates, the top surface deforms

* Corresponding author.
E-mail addresses: Kelsey.L.Dipietro.5@nd.edu (K.L. DiPietro), rhaynes@mun.ca (R.D. Haynes), whuang@ku.edu (W. Huang), a.lindsay@nd.edu (A.E. Lindsay), y920y782@ku.edu (Y. Yu).

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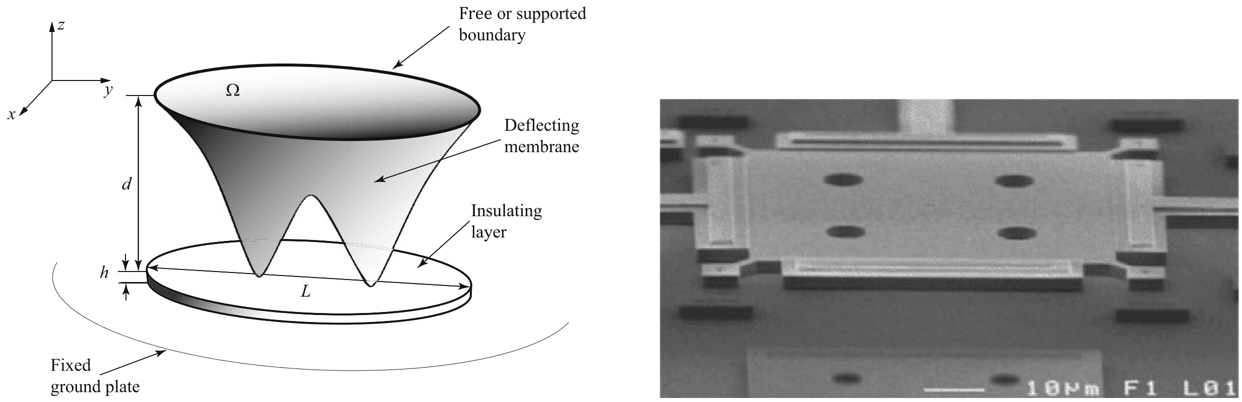
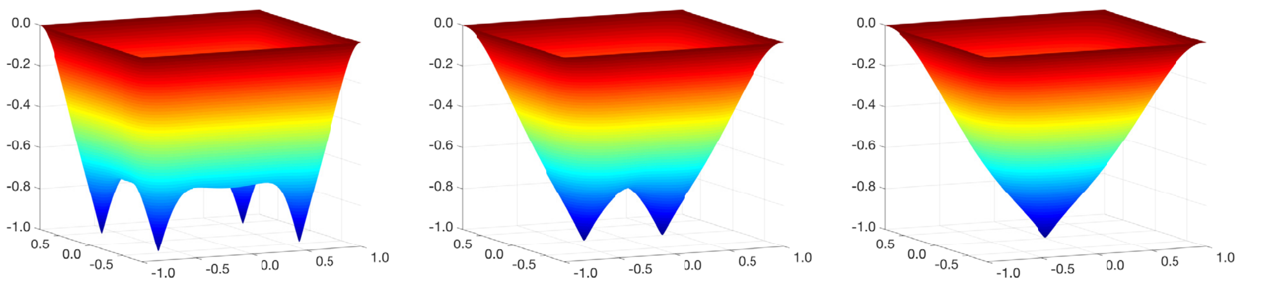


(a) Schematic diagram. (b) A MEMS device (source: [42])

Fig. 1. A MEMS device (right) and a schematic (left) around which models are formulated.



(a) $\varepsilon = 0.02$. (b) $\varepsilon = 0.068$. (c) $\varepsilon = 0.1$.

Fig. 2. Solutions $u(\mathbf{x}, t)$ of (1) at touchdown for $\varepsilon = 0.02, 0.068, 0.1$ in the rectangle $\Omega = (-1, 1) \times (-0.8, 0.8)$.

towards the substrate. In equation (1), the parameter ε quantifies the relative importance of electrostatic and elastic forces in the system. If the restorative elastic forces are too weak, the attractive Coulomb forces between the two surfaces will bring them into physical contact. This event, called *touchdown* or snap-through, can be useful or deleterious to operation, depending on the design of particular MEMS. The mechanism of the pull-in phenomenon has been studied extensively and many references can be found in the reviews [3,52].

The design and operation of MEMS can be aided by placing physical limiters or constraints at locations where contact between the two membranes is more likely [25]. These limiters can prevent damage to the device that could occur when the two surfaces meet. In addition, they allow for bistability in the system by creating stable large deflection configurations [26,27,30,32,40]. Therefore, it is important to know at which location(s) in Ω singularities can form. In the one-dimensional case with $\Omega = (-1, 1)$, equation (1) can form one singularity at the origin or two singularities located symmetrically about the origin, depending on the particular value of ε [29]. In the physically relevant two-dimensional scenario, the details of the geometry Ω and the dependence on the parameter ε combine to make the set of possible singularity locations much more complex [31,33].

Touchdown is a very rapid process in which energy is rapidly focused in small spatial regions of Ω. This process is manifested in the governing equations (1) by a finite time *quenching* singularity. The term quenching refers to the fact that $u(\mathbf{x}, t)$ is finite at the point of singularity while $u_t(\mathbf{x}, t)$ diverges as $t \to T$. Theoretical results on the quenching behavior of fourth-order parabolic equations such as (1) have established conditions under which quenching may occur [28,29], studied the local form of the profile near singularity [4,14,29,33] and given upper and lower estimates of the singularity time [13,29,39]. For reviews on the extensive literature on blow-up/quenching for parabolic PDEs, see [15] and references therein.

The aim of this paper is to explore, through numerical simulations and asymptotic analysis of (1) as $\varepsilon \to 0$, the potential set of locations at which touchdown may occur. In particular, we consider how the complex geometry and topology of MEMS devices, as seen in Fig. 1(b), influences the possible set of contact locations. We present an adaptive moving mesh strategy [24] for the solution of (1) which dynamically relocates the mesh points to provide resolution in the vicinity of forming singularities. An example of our method for the rectangular domain $\Omega = (-1, 1) \times (-0.8, 0.8)$ is shown in Fig. 2 which shows either 4, 2 or 1 contact points for different values of ε. The sensitive dependence of the contact set on the domain Ω and parameter ε will be explored with a geometric skeleton theory (Sec. 2) and adaptive numerical simulations (Sec. 3).