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# Stabilized conservative level set method

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This paper addresses one of the main challenges of the conservative level set method, namely the ill-conditioned behavior of the normal vector away from the interface. An alternative formulation for reconstruction of the interface is proposed. Unlike commonly used methods, which rely on a unit normal vector, the Stabilized Conservative Level Set (SCLS) makes use of a modified normal vector with diminishing magnitude away from the interface. With the new formulation, in the vicinity of the interface the reinitialization procedure utilizes compressive flux and diffusive terms only in normal direction with respect to the interface, thus, preserving the conservative level set properties, while away from the interface the directional diffusion mechanism automatically switches to homogeneous diffusion. The proposed formulation is robust and general. It is especially well suited for use with the adaptive mesh refinement (AMR) approaches, since for computational accuracy high resolution is only required in the vicinity of the interface, while away from the interface low resolution simulations might be sufficient. All of the results reported in this paper are obtained using the Adaptive Wavelet Collocation Method, a general arbitrary order AMR-type method, which utilizes wavelet decomposition to adapt on steep gradients in the solution while retaining a predetermined order of accuracy. Numerical solution for a number of benchmark problems has been carried out to demonstrate the performance of the SCLS method.



## 1. Introduction

Interfacial phenomena are quite common in fluid mechanics, e.g. bubbles and drops, waves, liquid films [1–4]. Surface or interface tracking methods can be roughly divided into two classes: a) interface tracking methods where the interfaces are represented explicitly, [e.g., 5,6] and b) level set methods, where the interfaces are represented implicitly by an isosurface or a level set of a function [e.g., 7,8]. The main advantage of level set methods compared to the interface tracking approaches is that the former handle topological changes such as merging and breaking more easily.

The level set approach was originally developed by Sethian [9]. Given a continuous level set function $f(\mathbf{x},t)$, the interface $\Gamma(t)$ is defined as

$$\Gamma(t) = \{\mathbf{x} \in R^d : f(\mathbf{x},t) = 0\}, \tag{1}$$

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which bounds a region $\Omega(t)$ of the domain $R^d$. A signed distance level set function $\phi(\mathbf{x},t)$ is defined as

$$\phi(\mathbf{x},t) = \mathcal{I}(\mathbf{x},t) \min_{\mathbf{y}\in\Gamma(t)} \|\mathbf{x}-\mathbf{y}\|_2 \ , \tag{2}$$

where $\mathcal{I}(\mathbf{x},t)$ is a sign function that indicates, whether $\mathbf{x}$ is inside or outside of the region $\Omega$

$$\mathcal{I}(\mathbf{x},t) = \begin{cases} 1, & \text{if } \mathbf{x} \in \Omega(t) \\ -1, & \text{if } \mathbf{x} \notin \Omega(t) \end{cases}, \tag{3}$$

while the zero level set defines the interface $\Gamma(t)$ itself.

The evolution equation of the level set function $\phi(\mathbf{x},t)$ is given by

$$\frac{\partial \phi}{\partial t} + \mathbf{u}\cdot\nabla\phi = 0, \tag{4}$$

where $\mathbf{u}$ is the velocity field. The unit normal vector can be computed as

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|}, \tag{5}$$

where for the level set function $\phi(\mathbf{x},t)$ the distance function condition

$$|\nabla\phi(\mathbf{x},t)| = 1 \tag{6}$$

should be satisfied. Note that the evolution of $\phi(\mathbf{x},t)$ according to (4) does not automatically guarantee the satisfaction of the condition (6). As a result, the level set function can be distorted and no longer represent the true distance function, thus, necessitating an introduction of a reinitialization procedure to reinforce the condition (6). For the reinitialization procedure Sussman et al. [10] proposed to solve the following Hamilton–Jacobi equation to steady state

$$\frac{\partial\tilde{\phi}}{\partial\tau} + \operatorname{sgn}(\phi)(|\nabla\tilde{\phi}| - 1) = 0, \tag{7}$$

$$\tilde{\phi}(\mathbf{x},0) = \phi(\mathbf{x},t), \tag{8}$$

where $\tau$ is the pseudo time, $\operatorname{sgn}(x) = 2H(x) - 1$ is the signum function, and $H(x)$ is the Heaviside function. In practice Eq. (7) needs to be solved for $\tau = O(\delta)$ to guarantee the condition (6) in the band of width $\delta$ around an interface $\Gamma(t)$. In fact, depending on the computational approach, the level set function may be defined only in a narrow band, if the information far from the interface is not used by the algorithm.

One of the main difficulties of the level set approach, especially in the context of incompressible flows, is the volume conservation, since traditional level set methods, even rewritten in divergence form for the divergence-free velocity field, do not guarantee the volume preservation. In order to overcome this difficulty, different methods have been proposed to improve the results of reinitialization procedure and enhance the volume conservation, the most influential of which are proposed by Sussman and Fatemi [11], Russo and Smereka [12], and Enright et al. [13]. Sussman and Fatemi suggested to add an additional constraint after solving the reinitialization Hamilton–Jacobi equation [11]. Russo and Smereka offered a modification to the discretization of the gradient of the signed distance function within one cell from the front using the information that $\phi$ is zero on the interface [12]. Enright et al. [13] have proposed a particle tracing method to overcome the difficulty associated with the accuracy of the level set method, where marker particles are used to reconstruct the level set function. For an in-depth discussion on different level set approaches, their advantages and disadvantages the reader is referred to Ref. [14].

In an attempt to construct a level set approach with built-in volume preserving properties Olsson and Kreiss [15] proposed a conservative level set method utilizing a regularized level set function

$$\psi(\mathbf{x},t) = \frac{1}{1+e^{-\frac{\phi(\mathbf{x},t)}{\delta}}} = \frac{1}{2}\left(\tanh\left(\frac{\phi(\mathbf{x})}{2\delta}\right)+1\right), \tag{9}$$

which changes between 0 and 1 in a narrow region with the thickness $O(\delta)$. The interface $\Gamma$ is represented by

$$\Gamma(t) = \{\mathbf{x}\in R^d : \psi(\mathbf{x},t) = 0.5\}. \tag{10}$$

Being formulated for incompressible flows, the evolution of $\psi(\mathbf{x},t)$ is governed by the advection equation written in conservative form

$$\frac{\partial\psi}{\partial t} + \nabla\cdot(\psi\mathbf{u}) = 0. \tag{11}$$

In the limit of interface thickness $\delta$ going to zero the integral $\int_{\Omega(t)}\psi(\mathbf{x},t)d\mathbf{x}$ approaches the volume of the region $\Omega(t)$, defined by $\int_{\Omega(t)} d\mathbf{x}$