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Energy dissipating flows for solving nonlinear eigenpair problems

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ABSTRACT

This work is concerned with computing nonlinear eigenpairs, which model solitary waves and various other physical phenomena. We aim at solving nonlinear eigenvalue problems of the general form $T(u) = \lambda Q(u)$. In our setting T is a variational derivative of a convex functional (such as the Laplacian operator with respect to the Dirichlet energy), Q is an arbitrary bounded nonlinear operator and λ is an unknown (real) eigenvalue. We introduce a flow that numerically generates an eigenpair solution by its steady state.
Analysis for the general case is performed, showing a monotone decrease in the convex functional throughout the flow. When T is the Laplacian operator, a complete discretized version is presented and anlalyzed. We implement our algorithm on *Korteweg and de Vries* (KdV) and *nonlinear Schrödinger* (NLS) equations in one and two dimensions. The proposed approach is very general and can be applied to a large variety of models. Moreover, it is highly robust to noise and to perturbations in the initial conditions, compared to classical Petiashvili-based methods.

1. Introduction

Nonlinear elliptic equations arise in various problems in physics, e.g. for stationary solutions of equations, such as *Bose–Einstein condensates* (BEC), *nonlinear Schrödinger* (NLS) and *Korteweg and de Vries* (KdV) [1]. In this work we focus on nonlinear problems of the form,

$$T(u) = \lambda Q(u), \tag{1}$$

where u is a function in a Banach space $\mathcal{U}$, and T and Q are (possibly) nonlinear operators. More specifically, we assume T to be a subgradient of a convex, proper, lower-semi-continuous functional J,

$$T(u) \in \partial_u J(u), \tag{2}$$

where $\partial_u J(u)$ denotes the subdifferential of $J(u)$. On the right-hand-side of (1), $Q : \mathcal{U} \to \mathcal{U}$ is a bounded (possibly) nonlinear operator. We refer to functions u which admit (1) as eigenfunctions, with a corresponding eigenvalue $\lambda \in \mathbb{R}$. Our aim is to find a pair $(u, \lambda) \in \mathcal{U} \times \mathbb{R}$ that admits (1), referred to as an *eigenpair*.

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Eigenpairs of nonlinear operators appear in various fields of science and engineering. Their analysis can provide deeper understanding and significant insights related to nonlinear systems. Nonlinear eigenvalue analysis is an active field of research, from both a theoretical and a computational perspective. In some nonlinear problems, such as [2] the underlying operators are linear but the dependency on λ is nonlinear. A recent review of such *nonlinear eigenvalue problems* (NEP) appear in [3]. These studies are part of a different branch of problems (not in the scope of this paper). We examine solutions which can be formulated by Eq. (1), where the eigenvalue is linearly dependent. We summarize below the main related studies.

1.1. Solitary waves as solutions of nonlinear eigenvalue problems

A pioneering work in this field, which was followed by many, is that of Petviashvili [4]. It was aimed at finding numerical approximations of stationary solutions for the Kadomtsev–Petviashvili equation with positive dispersion (KPI equation). The method, originally, was develop to obtain stationary solutions of wave equations of the form

$$-Mu + u^p = 0, \tag{3}$$

where M is a positive, self-adjoint operator and p is a constant. We note that M should be invertible, as the iterative procedure is based on its inversion. Conditions for the convergence of Petviashvili's method were established in [5]. The approach of Petviashvili was later generalized and applied to a family of nonlinear problems, such as [6–9]. However, all these algorithms assume M is invertible. Moreover, it is not aimed at finding eigenpairs, but at solving a more restricted problem. When casted within the formulation of (1), the eigenvalue is set to a unit value ($\lambda = 1$). Our proposed method is based on a forward flow, and hence M is not require to be invertible. Moreover, it is aimed at finding eigenpairs, of unknown λ. The resulting eigenpair is related to an initial condition, provided by the user, which can emerge from noisy experimental data, for instance.

Yang and Lakoba [7,10,11] generalized Petviashvili's iteration method, accelerated the inverse power method and used modified conjugate gradient to find solitary waves. In [12] it was suggested to combine the conjugate gradient method with accelerated inverse power method into a unified algorithm, which coincides with Petviashvili's method for small enough error. This method was shown to provide fast convergence rates. In our work we compare the numerical results to this method and to a modified version of it for adaptively computing eigenpairs. The focus of this paper is on the robustness of the methods, rather than on the convergence rate. We note that our forward flow requires considerably more iterations to converge, compared to algorithms based on inversion. However, it is much more stable and general.

1.2. Variational formulations of eigenpair problems

Eigenvalue problems are often analyzed theoretically and solved numerically based on energy minimization methods. Within a variational setting, it can be shown that an eigenpair is an extremum of a generalized Rayleigh quotient [13]. This extends in a natural manner to the linear case, where any eigenvector v of a Hermitian matrix A, admitting $Av = \lambda v$, is an extremal point of the associated Rayleigh quotient $R(v) = (v^* A v)/(v^* v) = \lambda$, where v^* is the conjugate transpose of v. The studies of [14–17] aim at finding a minimum (or a local minimum) of an energy functional associated to the eigenvalue problem. In [14,15] a constrained steepest descent is used for solving ground states of BEC. Alternative approaches, such as [16,17], are based on constrained energy minimization techniques with a suitable Lagrange function. In the above cases, both sides of the eigenvalue problem (1) should have an associated energy. This puts some limitations on the variety of problems that can be solved. In our work this restriction is relaxed (so only the operator T is associated with an energy term).

1.2.1. Eigenpairs associated with total variation

The *total variation* (TV) functional, $J_{TV} = \int |\nabla u(x)| dx$, has been thoroughly investigated in recent decades [18]. Since its introduction to the image processing field for denoising and deconvolution by [19], it has been used as an edge preserving regularizer for algorithms related to stereo imaging, optical flow, segmentation and many other computer vision tasks [20]. Eigenpairs associated with TV were investigated in [21] and [22]. It was shown that convex disk-like shapes are eigenfunctions of the nonlinear eigenvalue problem

$$T(u) = \lambda u, \;\; T(u) \in \partial_u J_{TV}(u),$$

where $\partial_u J_{TV}(u)$ is the subdifferential of TV. For smooth, non-vanishing gradient, we have $\partial_u J_{TV}(u) = -\mathrm{div}(\nabla u/|\nabla u|)$, which is the 1-Laplacian operator. In recent years, a theory of nonlinear transforms for one-homogeneous functionals has been formulated [23–25]. It is based on the analysis of nonlinear eigenvalue problems. The work of [26] proposed a flow for finding eigenfunctions of one-homogeneous functionals, and is described in more detail below. Numerical methods for finding p-Laplacian eigenpairs were proposed in [27] and [28]. Solutions of semilinear elliptic eigen-problems were presented in [1]. Ground states of generalized eigenvalue problems, which may involve also a smoothing kernel, were analyzed in [29]. An iterative algorithm for finding nonlinear eigenpairs by an extended inverse-power method was proposed by [30]. All of these methods are based partially or mostly on the fact that the eigenpair satisfies an extremum of the associated (generalized) Rayleigh quotient.

Our work generalizes the flow of [26]. It goes beyond the variational setting and allows the nonlinear operator Q in Eq. (1) to be very general.