Nonlinear Analysis 179 (2019) 237–253

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Solutions with moving singularities for equations of porous medium type

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A R T I C L E I N F O

Article history:
Received 27 June 2018
Accepted 30 August 2018
Communicated by Enzo Mitidieri

MSC:
35K65
35K67
35A01
35B40

Keywords:
Porous medium
Fast diffusion
Singular solution
Moving singularity

A B S T R A C T

We construct positive solutions of equations of porous medium type with a singularity which moves in time along a prescribed curve and keeps the spatial profile of singular stationary solutions. It turns out that there appears a critical exponent for the existence of such solutions.

1. Introduction

We study positive singular solutions of the following equation of porous medium type:

$$v_t = \Delta_y v^m, \qquad y \in \mathbb{R}^n \setminus \{\xi(t)\}, \quad t > 0, \tag{1.1}$$

where $m > 0$ and $\xi \in C^1([0,\infty);\mathbb{R}^n)$ is a given function. We consider (1.1) with the initial condition

$$v(y,0) = v_0(y), \qquad y \in \mathbb{R}^n \setminus \{\xi(0)\}. \tag{1.2}$$

We are interested in positive solutions that are singular at $\xi(t)$, that is,

$$v(y,t) \to \infty \qquad \text{as } y \to \xi(t), \quad t > 0.$$

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https://doi.org/10.1016/j.na.2018.08.016

For example, when $\xi \equiv 0$ and $n \geq 3$, (1.1) has a singular steady state given by

$$\tilde{v}(y) = K|y|^{-\frac{n-2}{m}}, \qquad y \in \mathbb{R}^n \setminus \{0\},$$

where K is an arbitrary positive constant. Another explicit solution for $\xi \equiv 0$ and

$$m_c := \frac{(n-2)_+}{n} < m < 1$$

is

$$V(y,t) := \left(\frac{ct}{|y|^2}\right)^{\frac{1}{1-m}}, \qquad c := 2m\left(\frac{2}{1-m} - n\right).$$

If $m = 1$, (1.1) is reduced to the linear heat equation. From [8] it follows that for every nonnegative solution v of

$$v_t = \Delta v, \qquad y \in \mathbb{R}^n \setminus \{\xi(t)\}, \quad t \in (0,T),$$

there is a nonnegative Radon measure M on $(0,T)$ such that

$$v_t = \Delta v + (\delta_0(y) \otimes M(t)) \circ \mathcal{T}_\xi \qquad \text{in } \mathcal{D}'(\mathbb{R}^n \times (0,T)).$$

Here δ_0 is the Dirac delta-function concentrated at $0 \in \mathbb{R}^n$ and $\mathcal{T}_\xi$ is a translation operator defined by $\mathcal{T}_\xi(\varphi)(y,t) := \varphi(y + \xi(t), t)$. Moreover, let $DM \in L^1_{\mathrm{loc}}((0,T))$ be the Radon–Nikodym derivative of the absolutely continuous part of M with respect to the Lebesgue measure. Then v satisfies

$$v(y,t) = DM(t)F(y - \xi(t)) + o(F(y - \xi(t)))$$

as $y \to \xi(t)$ for almost all $t \in (0,T)$, where F is the fundamental solution of Laplace's equation in $\mathbb{R}^n$. For the existence of singular solutions, see [7,18]. See also [8,9,15–17,19] for semilinear heat equations, and [10] for the Navier–Stokes system.

In [2], for $0 < m < 1$ and $n \geq 2$, one can find a complete classification of nonnegative solutions of $v_t = \Delta v^m$ in $\mathcal{D}'((\mathbb{R}^n \setminus \{0\}) \times (0,\infty))$ which are continuous in $\mathbb{R}^n \times [0,\infty)$ with values in $(0,\infty]$, unbounded at $y = 0$, and satisfy the initial condition

$$v(y,0) = 0, \qquad y \in \mathbb{R}^n \setminus \{0\}. \tag{1.3}$$

In some sense, these solutions are either of the same type as V or there is $\tau \in (0,\infty]$ such that they satisfy

$$v_t = \Delta v^m + \delta_0(y) \otimes M(t) \qquad \text{in } \mathcal{D}'(\mathbb{R}^n \times (0,\tau)),$$

for some positive Radon measure M, while they are of type V for $t > \tau$. If $m_c < m < 1$ and $M(t) = t^\sigma$ for some $\sigma \in [0, m/(1-m)]$, then the solution is of the self-similar form

$$v(y,t) := t^\alpha f(yt^{-\beta}), \qquad \alpha := \frac{2\sigma + 2 - n}{n(m-1)+2}, \quad \beta := \frac{m - \sigma(1-m)}{n(m-1)+2},$$

where f satisfies an elliptic equation, see [2]. For $0 < m < m_c$, $n \geq 3$, there are also self-similar solutions with a standing singularity, see [6,20].

If $m_c < m < 1$ then all weak solutions of $v_t = \Delta v^m$ with locally integrable initial data v_0 become immediately bounded and continuous, see [5]. On the other hand, in the same range $m_c < m < 1$, the strongly singular set of v_0 cannot shrink in time for extended continuous solutions, see [3]. Here the strongly singular set of v_0 is defined as the set of points at which v_0 is not locally integrable and an extended