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Irreversibility from staircases in symplectic embeddings

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ABSTRACT

We present an argument whose goal is to trace the origin of the macroscopically irreversible behavior of Hamiltonian systems of many degrees of freedom. We use recent flexibility and rigidity results of symplectic embeddings, quantified via the (stabilized) Fibonacci and Pell staircases, to encode the underlying breadth of the possible initial conditions, which alongside the multitude of degrees of freedom of the underlying system give rise to time-irreversibility.

1. Introduction

The origin of the macroscopic (time-) irreversibility observed in nature has been a topic of recurrent interest in Physics, since the late 19th century. The question it addresses is how does the macroscopically irreversible behavior arise, even though the underlying dynamics, in its Newtonian, Lagrangian, Hamiltonian etc. formulations, is time reversible. We are interested in classical, as opposed to quantum, behavior in this work, and we have in mind particle systems, even though a large part of this discussion can presumably be carried over to the Statistical Mechanics of fields.

The essence of the argument for irreversibility has been captured by L. Boltzmann's proposals [1,2] who, in our opinion, has laid out the main ideas that lead toward a resolution of this issue. However deep Boltzmann's arguments are, they have always been considered heuristic, waiting for a more rigorous justification, which the hope was, ergodic theory might be able to provide. This task is still unfinished though. The appearance of recent work on this issue such as [3,4] and the misunderstandings as pointed out in [4] in its proposed resolution point out toward its incomplete state of affairs, even today, and at the same time toward the conceptual and technical depth of this issue.

The relatively recent construction of a multitude of entropic functionals [5], and an independent but concurrent re-examination of the foundations of Statistical Mechanics [6], especially in the context of long-range interactions [7], is an additional incentive for a parallel investigation of the arguments on the origin of time-irreversibility, especially in the broader context of the dynamical foundations of the "non-classical", namely not the Boltzmann/Gibbs/Shannon (BGS) entropies.

In the conjectured dynamical framework of power-law entropies, we assumed in some of our previous works [8–11] that the non-additive behavior of systems described by the q-entropy (also known as "Tsallis entropy") may become manifest even for very few degrees of freedom. In the present work, we examine whether the time-irreversibility of Hamiltonian systems of many degrees of freedom can have any manifestation, even when one examines reduced systems of two effective degrees of freedom. This is a level of reductionism that is exactly opposite to the complex systems that the q-entropy claims

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to describe, but it may be more technically tractable and may provide some form of insight for the general case of Hamiltonian systems of many degrees of freedom, whose description is our ultimate goal.

In the present work, we do not use any particular entropic functional in our arguments. We rely, instead, on the underlying dynamical description of systems which are assumed to be modeled by autonomous Hamiltonians [12–14]. We rely on results mostly obtained during the current decade, some of the references for which are [15–34]. The present work can be seen as a physical application and interpretation of these results of symplectic geometry to aspects of Hamiltonian mechanics which may be pertinent to, and with a view toward, Statistical Physics. Many of the above results in symplectic geometry rely on and extend the foundational work of Gromov [35] Hofer–Zehnder, Ekeland–Hofer [14] etc., some of which were used in the closely related [36] to argue for time irreversibility from essentially the same perspective. The current work extends [36] which considered time irreversibility as a consequence of the symplectic non-squeezing theorem, but relies on the more recent mathematical developments stated in the above references. Unlike the afore-mentioned mathematical works which are rigorous, our arguments are hand-waving, attempting to provide a suggestive picture that may be pursued further in concrete models of physical significance, rather than firmly establishing generically applicable results.

Our conclusion in the current work is that the intricate pattern of flexibility and rigidity of symplectic embeddings quantified through the stabilized symplectic staircases of ellipsoids into balls, which express the behavior of sets of initial conditions of the symplectic flows, alongside the large number of degrees of freedom of the full/unreduced systems, can provide a plausible explanation for time-irreversibility, traces of which can be detected even in systems having two effective degrees of freedom.

In Section 2, we provide some background from symplectic geometry in order to make the presentation reasonably self-contained to our intended audience, and to set up the notation. Section 3 covers they key results about the symplectic non-squeezing theorem and symplectic capacities. Section 4 contains recent results from the literature on symplectic embeddings and their obstructions. In Section 5, we point out how the above concepts and results can be interpreted as the source of time-irreversibility for Hamiltonian systems and manifest themselves for systems having two effective degrees of freedom. Section 6 states some conclusions and posits some questions of interest to be tackled in the future.

2. Symplectic basics

There are several excellent sources for symplectic geometry and Hamiltonian dynamics these days. We have found parts of [12,14,37,38] to be very useful for foundational material. Moreover we have found the lucid and accessible work of M. de Gosson and collaborators [39–44] on the symplectic view of classical, semi-classical and quantum Physics to be a great motivation and help for this work. In addition, one may wish to consult [45,46] for applications of the symplectic non-squeezing theorem in Astrophysics and Celestial Mechanics. These papers are also very enlightening in making quite transparent and concrete some otherwise abstract concepts of symplectic geometry and in smoothing out terse aspects of some mathematical expositions. One might add to this their considerable originality and the potentially substantial impact of their results for many more branches of Physics.

2.1. Hamiltonian motivation

We consider physical systems which are modeled by autonomous (time-independent) Hamiltonians $H(x_1, \ldots, x_n, y_1, \ldots, y_n) : \mathcal{M} \to \mathbb{R}$ where (x_i, y_i), $i = 1, \ldots, n$ are (Hamiltonian-) conjugate coordinates parametrizing the $2n$-dimensional phase space $\mathcal{M}$ of the system. Without loss of generality, we will assume that $\mathcal{M} = \mathbb{R}^{2n}$, at least at this initial stage. The evolution of the system, in time t, is described by Hamilton's equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \qquad i = 1, \ldots, n \tag{1}$$

which can be rewritten, using the shorthand notation

$$z = (x_1, \ldots, x_n, y_1, \ldots, y_n) \tag{2}$$

as

$$\frac{dz}{dt} = -J_0 \, \nabla H(z) \tag{3}$$

where the $2n \times 2n$ matrix J_0 has the form

$$J_0 = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ 1_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{4}$$

where $0_{n\times n}$ and $1_{n\times n}$ stand for the zero and the unit (diagonal) $n \times n$ matrices. One can readily see that J_0 is antisymmetric, that $J_0^2 = -1_{2n\times 2n}$, and that J_0 rotates the coordinates of each 2-plane of canonically conjugate variables counterclockwise by $\pi/2$. The vector field

$$X_H = -J_0 \, \nabla H \tag{5}$$