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A new method for the approximation of integrals using the generalized Bernstein quadrature formula

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ABSTRACT

In the present paper, we establish the theoretical framework of a new method in order to approximate a definite integral of a given function by the generalized Bernstein quadrature formula. Some numerical examples will be given as support of the theoretical aspects. We want to highlight an applicative side of the Bernstein polynomials, in contrast to the well-known theory of the uniform approximation of the functions.

1. Introduction and auxiliary results

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The Bernstein operator [1] associated to any real-valued function $F : [0, 1] \to \mathbb{R}$, any $x \in [0, 1]$ and any $n \in \mathbb{N}$ is defined by

$$B_n(F; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} F\left(\frac{k}{n}\right). \tag{1}$$

Now, let $a, b \in \mathbb{R}$ be two real and finite numbers, such that $a < b$, for which we define the application l: $[0, 1] \to [a, b]$, given by $l(x) = a + x(b-a)$. Because the function l is bijective, it follows that l is invertible and $l^{-1} : [a, b] \to [0, 1]$, $l^{-1}(y) = \frac{y-a}{b-a}$. Using the notations $x = \frac{y-a}{b-a}$, $1 - x = \frac{b-y}{b-a}$, one can obtain the generalization of the Bernstein operator associated to any real-valued function $G : [a, b] \to \mathbb{R}$, $G := F \circ l^{-1}$, respectively to the knots $y = a + \frac{k(b-a)}{n}$. It follows

$$\begin{aligned} B_n(F; x) &= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} F\left(\frac{k}{n}\right) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{y-a}{b-a}\right)^k \left(\frac{b-y}{b-a}\right)^{n-k} \left(F \circ l^{-1}\right)\left(a + \frac{k(b-a)}{n}\right) \\ &= \frac{1}{(b-a)^n} \sum_{k=0}^{n} \binom{n}{k} (y-a)^k (b-y)^{n-k} G\left(a + \frac{k(b-a)}{n}\right) =: B_n^*(G; y). \end{aligned} \tag{2}$$

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Switching the variable y to x, we can write the generalization of the Bernstein operator associated to any real-valued function $G : [a, b] \to \mathbb{R}$, any $x \in [a, b]$ and any $n \in \mathbb{N}$, such that

$$B_n^*(G; x) = \sum_{k=0}^{n} p_{n,k}^*(x) G\left(a + \frac{k(b-a)}{n}\right) = \frac{1}{(b-a)^n} \sum_{k=0}^{n} \binom{n}{k} (x-a)^k (b-x)^{n-k} G\left(a + \frac{k(b-a)}{n}\right). \tag{3}$$

Remark 1. The relation (3) can be found also in the papers [3–7,9].

We define the application l_*: $[a, b] \to [0, 1]$, given by $l_*(x) = \frac{x-a}{b-a}$. Because the function l_* is bijective, it follows that l_* is invertible and $l_*^{-1} : [0, 1] \to [a, b]$, $l_*^{-1}(y) = a + y(b-a)$. Using the notations $\frac{x-a}{b-a} = y$, $\frac{b-a}{b-a} = 1 - y$, one can obtain the Bernstein operator associated to any real-valued function $F : [0, 1] \to \mathbb{R}$, $F := G \circ l_*^{-1}$, respectively to the knots $y = \frac{k}{n}$. It follows

$$\begin{aligned} B_n^*(G; x) &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x-a}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-k} G\left(a + \frac{k(b-a)}{n}\right) = \sum_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} \left(G \circ l_*^{-1}\right)\left(\frac{k}{n}\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} F\left(\frac{k}{n}\right) =: B_n(F; y). \end{aligned} \tag{4}$$

Switching the variable y to x, we get the Bernstein operator (1) associated to any real-valued function $F : [0, 1] \to \mathbb{R}$, any $x \in [0, 1]$ and any $n \in \mathbb{N}$. The Bernstein polynomials (1) opened a new era in the approximation theory starting with the year 1912, when Sergei Natanovich Bernstein presented his famous proof of the Weierstrass approximation theorem and continuing until today with thousands of interesting papers. Thanks to some important properties as the uniform approximation, the shape preservation and the variation diminishing, the Bernstein polynomials are indispensable tools in computer aided geometric design, as well as in other areas of mathematics. In the present paper, we want to highlight an applicative side of the Bernstein polynomials, in contrast to the well-known theory of the uniform approximation of the functions. In this respect, one example could be the approximation of various definite integrals, by using a quadrature formula based on the generalized Bernstein polynomials (3). In order to get a complete algorithm for this kind of approximation, we need some auxiliary results.

For any $n \in \mathbb{N}$, let $a \le x_0 < x_1 < \cdots < x_n \le b$ be some nodes. In many books on Numerical Analysis, divided differences for distinct nodes and any real-valued function $h : [a, b] \to \mathbb{R}$ are defined recursively

$$[x_0, x_1, \ldots, x_n; h] = \frac{1}{x_n - x_0}([x_1, \ldots, x_n; h] - [x_0, \ldots, x_{n-1}; h]). \tag{5}$$

Using the relation (5), the divided difference of the function h with respect to the distinct nodes x_0, x_1 is $[x_0, x_1; h] = \frac{1}{x_1 - x_0}(h(x_1) - h(x_0))$ and the divided difference of the function h with respect to the distinct nodes x_0, x_1, x_2 can be written

$$[x_0, x_1, x_2; h] = \frac{h(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{h(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{h(x_2)}{(x_2 - x_0)(x_2 - x_1)}. \tag{6}$$

Recently, Abel and Ivan [2] had an excellent idea to give a representation of the remainder in the Bernstein approximation formula

$$F(x) = B_n(F; x) + R_n(F; x), \tag{7}$$

for all functions defined on the interval [0, 1].

Theorem 1. *[2] For all $x \in (0, 1)$ the remainder of the Bernstein approximation formula (7) possesses the representation*

$$\begin{aligned} R_n(F; x) &= -\frac{x(1-x)}{n^2} \sum_{j=0}^{n-1} \sum_{i=0}^{n-1-j} p_{n-1-j,i}(x) \left[\frac{i+jx}{n}, \frac{i+(j+1)x}{n}, \frac{i+1+jx}{n}; F\right] \\ &= -\frac{x(1-x)}{n^2} \sum_{j=0}^{n-1} \sum_{i=0}^{n-1-j} p_{n-1-j,i}(x) \left[0, \frac{x}{n}, \frac{1}{n}; F(\cdot + y_{i,j})\right], \end{aligned} \tag{8}$$

where $y_{i,j} := \frac{i+jx}{n}$, $(j = 0, \ldots, n-1;\ i = 0, \ldots, n-1-j)$.

The focus of this paper is to show how can be approximated a definite integral of a given function by the generalized Bernstein quadrature formula. In order to reach this aim, some theoretical aspects are needed. We establish a representation of the remainder term in the generalized Bernstein approximation formula for arbitrary functions, as a convex combination of divided differences of second order on known nodes. We also get an upper bound estimation for the remainder term, when the approximated function possesses bounded divided differences of second order. According to our knowledge, an upper bound estimation for the remainder in the generalized Bernstein approximation formula was presented in [5], without giving a rigorous proof of it and seems to be wrong. Being motivated by this fact, in what follows we try to prove, respectively correct the recalled issues. Using the representation of the remainder term and the corrected upper bound