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Sobriety of quantale-valued cotopological spaces ☆

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Abstract

For each commutative and integral quantale, making use of the fuzzy order between closed sets, a theory of sobriety for quantale-valued cotopological spaces is established based on irreducible closed sets.

Keywords: Fuzzy topology; Quantale; Quantale-valued order; Quantale-valued cotopological space; Irreducible closed set; Sobriety

1. Introduction

A topological space X is sober if each of its irreducible closed subsets is the closure of exactly one point in X. Sobriety of topological spaces can be described via the well-known adjunction

$$\mathcal{O} \dashv \mathrm{pt}$$

between the category Top of topological spaces and the opposite of the category Frm of frames [10]. Precisely, X is sober if $\eta_X : X \longrightarrow \mathrm{pt}(\mathcal{O}(X))$ is a bijection (hence a homeomorphism), where η denotes the unit of the adjunction $\mathcal{O} \dashv \mathrm{pt}$.

In the classical setting, a topological space can be described in terms of open sets as well as closed sets, and we can switch between open sets and closed sets by taking complements. So, it makes no difference whether we choose to work with closed sets or with open sets. In the fuzzy setting, since the table of truth-values is usually a quantale, not a Boolean algebra, there is no natural way to switch between open sets and closed sets. So, it may make a difference whether we postulate topological spaces in terms of open sets or in terms of closed sets. An example in this regard is exhibited in [3,4].

The frame approach to sobriety of topological spaces makes use of open sets; while the irreducible-closed-set approach makes use of closed sets. Extending the theory of sober spaces to the fuzzy setting is an interesting topic in fuzzy topology. Most of the existing works focus on the frame approach; that is, to find a fuzzy counterpart of the category Frm of frames, then establish an adjunction between the category of fuzzy topological spaces and that of *fuzzy frames*. Works in this regard include Rodabaugh [26], Zhang and Liu [33], Kotzé [13,14], Srivastava and

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Khastgir [28], Pultr and Rodabaugh [21–24], Gutiérrez García, Höhle and de Prada Vicente [6], and Yao [30,31], etc. But, the irreducible-closed-set approach to sobriety of fuzzy topological spaces is seldom touched, except in Kotzé [13,14].

In this paper, making use of the fuzzy inclusion order between closed sets, we establish a theory of sobriety for quantale-valued topological spaces based on irreducible closed sets. Actually, this theory concerns sobriety of *quantale-valued cotopological spaces*. By a quantale-valued cotopological space we mean a "fuzzy topological space" postulated in terms of closed sets (see Definition 2.6). The term *quantale-valued topological space* is reserved for "fuzzy topological space" postulated in terms of open sets (see Definition 3.16).

It should be noted that in most works on fuzzy frames, the table of truth-values is assumed to be a complete Heyting algebra (or, a frame), even a completely distributive lattice sometimes. But, in this paper, the table of truth-values is only assumed to be a commutative and integral quantale. Complete Heyting algebra, BL-algebras and left continuous t-norms, are important examples of such quantales.

The contents are arranged as follows. Section 2 recalls basic concepts about quantale-valued ordered sets and quantale-valued $\mathcal{Q}$-cotopological spaces. Section 3, making use of the quantale-valued order between closed sets in a $\mathcal{Q}$-cotopological space, establishes a theory of sober $\mathcal{Q}$-cotopological spaces based on irreducible closed sets. In particular, the sobrification of a stratified $\mathcal{Q}$-cotopological space is constructed. The last section, Section 4, presents some interesting examples in the case that $\mathcal{Q}$ is the unit interval [0, 1] coupled with a (left) continuous t-norm.

2. Quantale-valued ordered sets and quantale-valued cotopological spaces

In this paper, $\mathcal{Q} = (Q, \&)$ always denotes a commutative and integral quantale, unless otherwise specified. Precisely, Q is a complete lattice with a bottom element 0 and a top element 1, & is a binary operation on Q such that $(Q, \&, 1)$ is a commutative monoid and $p \& \bigvee_{j\in J} q_j = \bigvee_{j\in J} p\&q_j$ for all $p \in Q$ and $\{q_j\}_{j\in J} \subseteq Q$.

Since the semigroup operation & distributes over arbitrary joins, it determines a binary operation → on Q via the adjoint property

$$p\&q \le r \iff q \le p \to r.$$

The binary operation → is called the *implication*, or the *residuation*, corresponding to &.

Some basic properties of the binary operations & and → are collected below, they can be found in many places, e.g. [2,27].

Proposition 2.1. *Let $\mathcal{Q}$ be a quantale. Then*

(1) $1 \to p = p$.
(2) $p \le q \iff 1 = p \to q$.
(3) $p \to (q \to r) = (p\&q) \to r$.
(4) $p\&(p \to q) \le q$.
(5) $\left(\bigvee_{j\in J} p_j\right) \to q = \bigwedge_{j\in J}(p_j \to q)$.
(6) $p \to \left(\bigwedge_{j\in J} q_j\right) = \bigwedge_{j\in J}(p \to q_j)$.

We often write $\neg p$ for $p \to 0$ and call it the *negation* of p. Though it is true that $p \le \neg\neg p$ for all $p \in Q$, the inequality $\neg\neg p \le p$ does not always hold. A quantale $\mathcal{Q}$ is said to satisfy the *law of double negation* if

$$(p \to 0) \to 0 = p,$$

i.e., $\neg\neg p = p$, for all $p \in Q$. A commutative and integral quantale that satisfies the law of double negation is also called a complete regular residuated lattice sometimes.

Proposition 2.2. ([2]) *Suppose that $\mathcal{Q}$ is a quantale that satisfies the law of double negation. Then*

(1) $p \to q = \neg(p\&\neg q) = \neg q \to \neg p$.
(2) $p\&q = \neg(q \to \neg p) = \neg(p \to \neg q)$.
(3) $\neg(\bigwedge_{i\in I} p_i) = \bigvee_{i\in I} \neg p_i$.