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On a topological fuzzy fixed point theorem and its application to non-ejective fuzzy fractals

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Abstract

A topological fuzzy fixed point theorem is given, when generalizing and improving the main result by Diamond, Kloeden and Pokrovskii (1997) [1]. Apart from the sole existence, a weak local stability property, called non-ejectivity in the sense of Browder, of fuzzy fixed points is established. This theorem is then applied for obtaining non-ejective fuzzy fractals. An alternative approach via the Knaster–Tarski theorem is also presented.

Keywords: Fuzzy fixed points; Absolute retracts; Hilbert cube; Multivalued maps; Non-ejectivity; Fuzzy fractals

1. Introduction

The main stimulation of the present note comes from the article [1], where the following topological fuzzy fixed point theorem was presented:

Theorem 1. *Let (X, d) be a compact metric space and $\mathcal{D}(X)$ be the totality of fuzzy sets $u\colon X \to [0, 1]$ which are upper semicontinuous (in the single-valued sense). Let, furthermore, $D_\delta(X), \delta \in (0, 1]$, be the class of all $u \in \mathcal{D}(X)$ such that $\max_{x\in X} u(x) \geq \delta$. Then the metric spaces $(\mathcal{D}(X), \mathrm{d_E})$ and $(\mathcal{D}_\delta(X), \mathrm{d_E})$, for $\delta \in (0, 1]$, where $\mathrm{d_E}$ stands for the endograph metric, each have the fixed point property, i.e. that any continuous mapping $f^*\colon \mathcal{D}(X) \to \mathcal{D}(X)$, resp. $f^*\colon \mathcal{D}_\delta(X) \to \mathcal{D}_\delta(X), \delta \in (0, 1]$, has at least one fixed point.*

Our purpose here is three-fold:

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(i) to extend Theorem 1, for $\mathcal{D}(X)$, that a basic space X need not be compact, provided the mapping f in X inducing f^* is compact,[2]
(ii) to improve Theorem 1, for $\mathcal{D}(X)$, in the sense that, under the assumptions related to (i) and the existence of an accumulation point of X, there always exists a non-ejective fuzzy fixed point,
(iii) to apply our generalized and improved fixed point theorem to non-ejective fuzzy fractals.

Each of the above purposes requires at least a brief explanation. Nevertheless, before making it, we must clarify the usage of an appropriate metric in $\mathcal{D}(X)$, resp. $\mathcal{D}_\delta(X)$, $\delta \in (0, 1]$, in Theorem 1, in order to avoid a possible misunderstanding. In [1], the authors speak rather incorrectly about the sendograph metric which was relevantly criticised and supplied by a counter-example in [2]. On the other hand, they used in fact the correct endograph metric. Therefore, despite the wrong name, their result is reliable.

Because of a desired application of Theorem 1 and its generalization presented in this paper, to fuzzy fractals, it will be advantageous to show that the (compact) maps f in the basic space X inducing $f^*\colon \mathcal{D}(X) \to \mathcal{D}(X)$ can be multivalued (see Lemma 2 below). Furthermore, the required existence of an accumulation point of X in (ii) can be omitted (see Section 5), when proceeding alternatively without the usage of the Browder-type fixed point theorems from [4] and [5]. Theorem 1 is a nice example of a topological (Schauder-like) fixed point theorem. That is also why we call the fractals obtained in this way, i.e. via topological fixed point theorems, as *topological*[3] *(fuzzy) fractals* (cf. [8,9]). All the fuzzy fractals determined by means of fixed points were received up to now exclusively in the frame of a metric (Banach-like) fixed point theory (see e.g. [10–13], and the references therein). Therefore, the application of theorems like Theorem 1 is a novelty in the theory of fuzzy fractals.

2. Preliminaries

Definition 1. A metric space (X, d) has *the fixed point property* if any continuous mapping $f\colon X \to X$ has at least one fixed point.

Let us also recall the notions of an absolute retract and the Hilbert cube.

Definition 2. Let X be a metric space and $A \subset X$ be its closed subset. We say that A is a *retract* of X if there is a continuous function $r\colon X \to A$ such that r restricted to A is the identity. A metric space X is called *an absolute retract* (AR) if it is a retract of every metrizable space Y containing X as a closed subset.

Definition 3. Let $Q = \prod_{i=0}^{\infty}[0, 1]$ be equipped with the metric $\mathrm{d_Q}$, where

$$\mathrm{d_Q}(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}.$$

Any homeomorphic image of the metric space $(Q, \mathrm{d_Q})$ is called the *Hilbert cube.*

Proposition 1. *(see e.g. [14,15]) Every compact absolute retract has the fixed point property. In particular, a homeomorphic image of the Hilbert cube has the fixed point property.*

In Section 5, we will also use the following notion of contractibility.

Definition 4. We say that a nonempty set $A \subset X$ of a metric space (X, d) is *contractible* if there exist a point $x_0 \in A$ and homotopy $h\colon A \times [0, 1] \to A$ such that $h(x, 0) = x$ and $h(x, 1) = x_0$, for every $x \in A$.

Let (X, d) be a metric space. We denote by $\mathbb{C}(X)$ and $\mathbb{K}(X)$ the spaces of nonempty, closed subsets and nonempty, compact subsets of X, respectively. Both these spaces are, as usually, equipped with the Hausdorff metric, i.e.

[2] This can be easily shown by means of the statements, e.g. in [2, Property 3.4.2] and [3], to be equivalent with a fuzzified f^* being compact.
[3] Unlike e.g. in [6,7], where the authors understand by topological fractals those obtained via the Edelstein metric fixed point theorem, up to homeomorphisms.

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