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Nonlinear three-dimensional diffusion models of porous medium in the presence of non-stationary source or absorption and some exact solutions

Yu.A. Chirkunov *, Yu.L. Skolubovich

Novosibirsk State University of Architecture and Civil Engineering (Sibstrin), 113 Leningradskaya st., 630008, Novosibirsk, Russia

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ABSTRACT

We study a general three-dimensional nonlinear diffusion model of porous medium with non-stationary source or absorption. We found nine basic models of the original model of the porous medium with non-stationary source or absorption, having different symmetry properties. For the model, admitting the widest group Lie of the transformations we found all invariant submodels. We found explicitly all essentially distinct invariant solutions describing invariant submodels of rank 0 of this model. In particular, we obtained the solutions, which we called "a layered circular pie", "a layered spiral pie", " layered plane pie" and "a layered spherical pie". The solution "a layered circular pie" describes a motion of the liquid or gas in a porous medium, for which at each fixed moment of a time at all points of each circle from the family of concentric circles a pressure is the same. The solution "a layered spiral pie" describes a motion of the liquid or gas in a porous medium, for which at each fixed moment of a time at all points of each logarithmic spiral, from the obtained family of logarithmic spirals a pressure is the same . The solution "a layered spherical pie" describes a motion of the liquid or gas in a porous medium, for which at each fixed moment of a time at all points of each sphere , from the family of concentric spheres a pressure is the same. A set of the solutions "a layered circular pie", "a layered spiral pie" and "a layered spherical pie" contains the solutions describing a distribution of the pressure in a porous medium after a point blast or a point hydraulic shock. Also this set contains the solutions describing a stratified with respect to the pressure a motion of liquid or gas in a porous medium, with a very high pressure at infinity in a presence of a very strong absorption at a point. The solution "a layered plane pie" describes a motion of the liquid or gas in a porous medium, for which at each fixed moment of a time at all points of each plane, from the family of parallel planes a pressure is the same. A set of the solutions "a layered plane pie" contains the solutions describing a motion of the liquid or gas in a porous medium with a very high pressure near a fixed plane in a presence of a very strong absorption at infinity. Also this set contains the solutions describing a motion of the liquid or gas in a porous medium with a very high pressure at infinity in a presence of a very strong absorption on a fixed plane. The obtained results can be used to study to describe the processes associated with a underground fluid or gas flow, with water filtration, with the engineering surveys in the construction of the buildings, and also with shale oil and gas production.

1. Introduction

There are a number of physical applications in which the model of porous medium is used by a natural way. First of all, this model is used to describe the processes associated with a underground fluid or gas flow, with water filtration, with the engineering surveys in the construction of the buildings, and also with shale oil and gas production. The study of fluid and gas motion in porous media within the framework of classical models does not always adequately describe real processes. This is due to the fact that these models do not take into account the presence of non-stationary source or absorption (see, for example, [1–6] and its references). For a more adequate description of the real processes, it is necessary to obtain and study new more complex three-dimensional models with nonstationary source or absorption.

Many mathematical models of physics and continuum mechanics are formulated in the form of linear and quasi-linear differential equations. Mathematical model is a description of the real scheme by mathematical language. The symmetry analysis of the equations of the models of physics and mechanics of continuous media is one of the most effective ways to obtain quantitative and qualitative characteristics of the physical processes. For example, in [7] the structure of symmetry groups

* Corresponding author.
E-mail addresses: chr101@mail.ru (Yu.A. Chirkunov), skolubovich@sibstrin.ru (Yu.L. Skolubovich).

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of some mathematical models of continuous medium mechanics was studied in order to elucidate the physical meaning of the admitted symmetries. The main task of the symmetry analysis of differential equations is to study the set of the solutions of these equations. All algorithms of the symmetry analysis are the preparation for achieving this purpose. The modern concept of the symmetry analysis is understood as the fullest using of the group of transformations admitted by the equations of model primarily to obtain and research the exact solutions. Exact solutions allow us to describe the specific physical processes. Exact solutions can be used as test solutions in numerical calculations, which perform in the studies of the real processes. Exact solutions allow us to assess the degree of adequacy of a given mathematical model to the real physical processes, after carrying out experiments corresponding to these decisions, and estimating the deviations that arise.

The main object of our study in this paper is a general three-dimensional nonlinear diffusion model of porous medium with non-stationary source or absorption, which is described by an equation:

$$p_t = \Delta\Phi(p) - f(t)(p+\beta), \tag{1}$$

where $p = p(t,\boldsymbol{x})$ is a pressure, t is a time, $\boldsymbol{x} = (x,y,z) \in R^3, \Delta = \partial_x^2+\partial_y^2+\partial_z^2$, β is an arbitrary constant, $\Phi(p)$ defines a nonlinearity of the process, $f(t)$ defines nonstationary nature of the source or absorption. $\Phi(p)$ and $f(t)$ are any functions, which are determined empirically. A case $f(t) < 0$ corresponds to the presence of a source. A case $f(t) > 0$ corresponds to the presence of an absorption. The functions $\Phi(p)$ and $f(t)$ satisfy to the condition

$$\Phi'(p)(p)f'(t) \neq 0. \tag{2}$$

This condition means that the process is nonlinear and the source or absorption is nonstationary.

2. Basic models

Without loss of generality we assume that $\beta = 0$. This can be achieved by replacing

$$\bar{p} = p+\beta, \bar{\Phi}(\bar{p}) = \Phi(\bar{p}-\beta).$$

The Eq. (1) takes a form (the line above the letters is not written)

$$p_t = \Delta\Phi(p) - f(t)\,p. \tag{3}$$

We study the group properties of this equation.

To obtain all having different symmetry properties basic models of the general model (3), we will fulfill group classification of Eq. (3) with condition (2). With a help of the algorithm proposed in [8,9] we will solve the problem of the group classification of this equation. In contrast to the classical algorithm presented in [8], this algorithm, firstly, avoids significant analytical difficulties associated with the analysis of the classification equations arising in the application of the algorithm from [10]; secondly, it significantly reduces the number of calculations. This algorithm was successfully used in [11–18] for group classification of various equations of mechanics and mathematical physics.

An arbitrary element of this system is $\boldsymbol{f} = (\Phi(p), f(t), \beta)$. Structure equations of an arbitrary element are written as follows:

$$\Phi_t = 0,\ \Phi_{\boldsymbol{x}} = \boldsymbol{0},\ f_{\boldsymbol{x}} = \boldsymbol{0},\ f_p = 0,\ \beta_t = 0,\ \beta_{\boldsymbol{x}} = \boldsymbol{0},\ \beta_p = 0. \tag{4}$$

The operator of generalized equivalence transformations of the Eq. (3) is defined as

$$\xi^0(t,\boldsymbol{x},p)\,\partial_t + \boldsymbol{\xi}(t,\boldsymbol{x},p)\,\partial_{\boldsymbol{x}} + \eta(t,\boldsymbol{x},p)\,\partial_p + \boldsymbol{\zeta}(t,\boldsymbol{x},p,\boldsymbol{f})\cdot\partial_{\boldsymbol{f}},$$

where $\xi^0, \boldsymbol{\xi} = (\xi^1,\xi^2,\xi^3), \eta, \boldsymbol{\zeta} = (\zeta^1,\zeta^2,\zeta^3)$ are smooth functions of their variables.

The condition of invariance of the manifold determined by Eqs. (3), (4) to this operator, with allowance [8,9] for the rule of extension of this operator after splitting in terms of parametric derivatives yields a system of the equations determining the generalized equivalence transformations of Eq. (3) and the specializations of the arbitrary element $\boldsymbol{f}$.

Solutions of this overdetermined system are all specializations of the arbitrary element and the corresponding equivalence transformations of Eq. (3). These equivalence transformations form the set of generalized equivalence transformations of Eq. (3). For the Eq. (3) the set of the generalized equivalence transformations of this equation coincides with the group of its universal equivalence transformations. For the specializations of the arbitrary element we study the action of the group of equivalences of Eq. (3) with this arbitrary element or, more exactly, the action of the factor-group of this group of equivalences by the kernel of the main groups of Eq. (3) on Eq. (3) with this arbitrary element. As a result of this action, equivalent equations are formed. To find all non-equivalent equations, we construct an optimal system of subgroups for the considered group of equivalences or, more exactly, for the factor-group of this group of equivalences by the kernel of the main groups of Eq. (3). The equivalence transformations acting on $\boldsymbol{f}$ identically form the kernel of the main groups of Eq. (3) with this arbitrary element $\boldsymbol{f}$, i.e., they are admitted by Eq. (3) for all elements $\boldsymbol{f}$ possessing the considered arbitrariness. In addition to the kernel of the main groups, Eq. (3) admits each subgroup of the group of equivalences under the condition that this subgroup acts on the element $\boldsymbol{f}$ identically. For each subgroup of the constructed optimal system of subgroups, the element $\boldsymbol{f}$ is specified under the condition that this subgroup acts on the element $\boldsymbol{f}$ identically. We formulate the final results of the group classification of Eq. (3) under condition (2). In all the subsequent formulas the prime denotes derivatives with respect to the corresponding variable.

• The kernel of the main groups of Eq. (3) is generated by the operators

$$\boldsymbol{X} = (X_1,X_2,X_3) = \partial_{\boldsymbol{x}}, \quad \boldsymbol{Z} = (Z_1,Z_2,Z_3) = \boldsymbol{x}\times\partial_{\boldsymbol{x}}. \tag{5}$$

• For

$$f = \pm\frac{1}{t}, \quad \Phi'(p) \neq 0 \tag{6}$$

the main group of Eq. (3) is generated by the operators (5) and operator

$$R_1 = 2t\partial_t + \boldsymbol{x}\cdot\partial_{\boldsymbol{x}}.$$

• For

$$f = -\alpha\left(\ln h_1{}'(t)\right)',\ \ h_1{}'(t) > 0,\ \ \alpha\beta\left(\ln h_1{}'(t)\right)'' \neq 0,$$
$$\Phi = \begin{cases} (p+\beta)^{\frac{\alpha+1}{\alpha}}, & \alpha \neq -1, \\ \ln(p+\beta), & \alpha = -1, \end{cases} \tag{7}$$

where $h_1(t)$ is a solution of the equation

$$\left(\ln h_1{}'(t)\right)'' = \frac{h_1{}''(t)}{h_1(t)}\left(2\alpha\gamma - (\alpha+1)\left(\frac{1}{\left(\ln h_1(t)\right)'}\right)'\right)$$
$$(\gamma \text{ is an arbitrary constant}), \tag{8}$$

the main group of Eq. (1) is generated by the operators (5) and operator

$$X_4 = X_4(\gamma) = \frac{1}{\ln h_1{}'(t)}\partial_t + \gamma\boldsymbol{x}\cdot\partial_{\boldsymbol{x}} + \alpha\left(2\gamma - \left(\frac{1}{\ln h_1{}'(t)}\right)'\right)p\partial_p.$$

In particular, at $\gamma = 0, \alpha \neq -1$ operator $X_4(0)$ takes a form

$$X_4(0) = \frac{1}{\ln h_1{}'(t)}\partial_t - \alpha\left(\frac{1}{\ln h_1{}'(t)}\right)'p\partial_p,$$

where $h_1(t)$ is implicitly determined by a quadrature

$$\int\left(a_1h_1^{-\alpha} + a_2\right)^{\frac{1}{\alpha}}dh_1 = t + a_3\ (a_1,a_2,a_3 \text{ are arbitrary constants}). \tag{9}$$

• For

$$f = -\alpha\left(\ln h_2{}'(t)\right)', h_2{}'(t) > 0,\ \ \alpha\beta\left(\ln h_2{}'(t)\right)'' \neq 0,$$
$$\Phi = \begin{cases} (p+\beta)^{\frac{\alpha+1}{\alpha}}, \alpha \neq -1, \\ \ln(p+\beta), \alpha = -1, \end{cases} \tag{10}$$