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On the essential spectrum of elliptic differential operators

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A B S T R A C T

Let $\mathcal{A}$ be a C^*-algebra of bounded uniformly continuous functions on a finite dimensional real vector space X such that $\mathcal{A}$ is stable under translations and contains the continuous functions that have a limit at infinity. Denote $\mathcal{A}^\dagger$ the boundary of X in the character space of $\mathcal{A}$. Then to each operator A in the crossed product $\mathcal{A} \rtimes X$ one may naturally associate a family of bounded operators $A_\varkappa$ on $L^2(X)$ indexed by the characters $\varkappa \in \mathcal{A}^\dagger$. We show that the essential spectrum of A is the union of the spectra of the operators $A_\varkappa$. The applications cover very general classes of singular elliptic operators.

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1. Introduction

1.1. Elliptic algebra

Let $X = \mathbb{R}^d$ and $L^2 = L^2(X)$. We denote $\mathscr{B} = \mathscr{B}(X)$ the algebra of bounded operators on L^2 and $\mathscr{K} = \mathscr{K}(X)$ that of compact operators. Let $\mathscr{B}_{\rm loc}$ be the space $\mathscr{B}$ equipped with the *local norm topology* defined by the family of seminorms $\|A\|_\theta = \|A\theta(q)\|$ where $\theta \in C_0(X)$ (continuous functions which tend to zero at infinity) and $\theta(q)$ means multiplication by θ. Clearly this topology is metrizable and finer than the strong operator topology. If $\theta(x) > 0\ \forall x$ then $\|\cdot\|_\theta$ is a norm on $\mathscr{B}$ which on bounded subsets defines the local norm topology. If $(A_s)_s$ is a sequence which converges in $\mathscr{B}_{\rm loc}$ to A, we say that the sequence is *locally norm convergent* and write $\text{u-lim}_s A_s = A$.

If $a \in X$ then $\mathrm{e}^{\mathrm{i}aq}$ and $\mathrm{e}^{\mathrm{i}ap}$ are the unitary operators on L that act as follows:

$$(\mathrm{e}^{\mathrm{i}aq}u)(x) = \mathrm{e}^{\mathrm{i}ax}u(x) \quad \text{and} \quad (\mathrm{e}^{\mathrm{i}ap}u)(x) = u(x+a). \tag{1.1}$$

We also use an alternative notation for the translation by $a \in X$ of a function, namely $\tau_a(\varphi)(x) = \varphi(a+x)$, and extend it to operators: $\tau_a(A) \doteq \mathrm{e}^{\mathrm{i}ap}A\mathrm{e}^{-\mathrm{i}ap}$ for $A \in \mathscr{B}$. The *elliptic algebra of* X (the name will be justified page 3) is defined by

$$\mathscr{E} = \{A \in \mathscr{B} \mid \lim_{a\to 0} \|(\mathrm{e}^{\mathrm{i}ap} - 1)A^{(*)}\| = 0,\ \lim_{a\to 0} \|\mathrm{e}^{-\mathrm{i}aq}A\mathrm{e}^{\mathrm{i}aq} - A\| = 0\}. \tag{1.2}$$

The notation $A^{(*)}$ means that the relation must hold for both A and A^*. Clearly $\mathscr{E}$ is a C^*-algebra. $\mathscr{E}_{\rm loc}$ is the set $\mathscr{E}$ equipped with the local norm topology inherited from $\mathscr{B}_{\rm loc}$.

Our main result requires more formalism but we can state right now the simplest particular case, which does not require any C^*-algebra background. We denote $X^\dagger$ the set of all ultrafilters finer than the Fréchet filter on X. We denote $\mathrm{Sp}(A)$ the spectrum and $\mathrm{Sp}_{\rm ess}(A)$ the essential spectrum of an operator A.

Theorem 1.1. *If $A \in \mathscr{B}$ then the limit $\text{u-lim}_{x\to\varkappa} \tau_x(A) \doteq A_\varkappa$ exists $\forall \varkappa \in X^\dagger$ and*

$$\mathrm{Sp}_{\rm ess}(A) = \bigcup\nolimits_{\varkappa\in X^\dagger} \mathrm{Sp}(A_\varkappa). \tag{1.3}$$

Remark. That the convergence holds locally in norm is important for the proof of the theorem. This type of convergence has been used in [19], see for example (4.24) there.

For the convenience of the reader, we recall in an appendix §5 some facts concerning filters and ultrafilters and also reformulate Theorem 1.1 in terms of sequences, as in [20,24,25] for example. We mention that ultrafilters have been first used in this context in [11, Th. 4.1] and then in [27]. In §5 we will see that ultrafilters play a quite natural role in the theory.

In §4.1 we will extend this result to the unbounded operators whose resolvent belongs to $\mathscr{E}$. We mention a simpler result in the self-adjoint case. Note that non-densely defined self-adjoint operators could appear as limits. For example, quite often $H_\varkappa = \infty$ where ∞ is the operator with $\{0\}$ as domain and 0 as resolvent. Clearly, H has purely discrete spectrum, i.e. $\mathrm{Sp}_{\rm ess}(H) = \emptyset$, if and only if $H_\varkappa = \infty$ for all $\varkappa$. See also §4.2.8.

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