Applied Mathematics and Computation 341 (2019) 215–228

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A wavelet approach for solving multi-term variable-order time fractional diffusion-wave equation

Mohammad Hossein Heydari[a,*], Zakieh Avazzadeh[b], Malih Farzi Haromi[a]

[a] Department of Mathematics, Shiraz University of Technology, Shiraz, Iran
[b] School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China

ARTICLE INFO

Keywords:
Multi-term variable-order time fractional diffusion-wave equation (M-V-TFD-E)
Chebyshev wavelets (CWs)
Operational matrix of variable-order Fractional derivative (OMV-FD)
Collocation method
Tau method

ABSTRACT

We firstly generalize a multi-term time fractional diffusion-wave equation to the multi-term variable-order time fractional diffusion-wave equation (M-V-TFD-E) by the concept of variable-order fractional derivatives. Then we implement the Chebyshev wavelets (CWs) through the operational matrix method to approximate its solution in the unit square. In fact, we apply the operational matrix of variable-order fractional derivative (OMV-FD) of the CWs to derive the unknown solution. We proceed with coupling the collocation and tau methods to reduce M-V-TFD-E to a system of algebraic equations. The important privilege of method is handling different kinds of conditions, i.e., initial-boundary conditions and Dirichlet boundary conditions, by implementing the same techniques. The convergence and error estimation of the CWs expansion in two dimensions are theoretically investigated. We also examine the applicability and computational efficiency of the new scheme through the numerical experiments.

1. Introduction

In recent years, many subjects in fractional calculus have been developed in various fields from pure mathematical theory to applied sciences such as modeling of heat transfer in heterogeneous media [1], modelling of the ultracapacitor and beam heating [2] and etc. These applications are mainly due to the fact that many physical systems are involving with fractional order dynamics and their behaviors are governed by fractional differential equations (FDEs) [3]. The significant importance of using FDEs is describing the non-local property [4], which means the current state and all its previous states affect on the next state of a dynamical system. We remind that difficulty in obtaining analytical solutions is an essential issue about fractional calculus problems. Therefore, numerical and approximation methods are commonly proposed to obtain approximate solutions for this kind of problems e.g. [5–21].

Presently, the variable-order fractional calculus as an development of the classical fractional calculus is getting more popular. From this perspective, the order of fractional integrals and derivatives are possibly functions of space, time and etc. With this approach, the non-local property is more evident and its wide applications have been found in physics [22–24], mechanics [25] for modeling of linear and nonlinear viscoelasticity oscillators [26,27], signal processing [28] and optimization control [29]. Also, some other physical models expressed via variable-order fractional derivatives can be found

* Corresponding author.
E-mail addresses: heydari@sutech.ac.ir, heydari@stu.yazd.ac.ir (M.H. Heydari), z.avazzadeh@njnu.edu.cn (Z. Avazzadeh), m.farzi@sutech.ac.ir (M.F. Haromi).

https://doi.org/10.1016/j.amc.2018.08.034
0096-3003/

in [30–32]. Some useful advantages and benefits of using variable-order fractional derivatives instead of the constant-order fractional derivatives have been discussed in [33]. In addition, some theoretical discussions about existence and uniqueness of solutions have been pointed in [34–36].

In spite of the importance of variable-order fractional functional equations in modelling of real phenomena, finding their analytical solutions is very costly process while applying the numerical methods are more reasonable [37–44]. In [20], the authors proposed the method of approximate particular solutions for solving constant- and variable-order fractional diffusion models. Recently, spectral methods have successfully been used to solve many types of V-FDEs due to their flexibility of implementation over finite and infinite domains. Moreover, the exponential convergence of these methods leads to high level of accuracy in practice. This property of spectral methods is usually called the "spectral accuracy" [45]. In [46,47], spectral methods through the shifted Jacobi polynomials are proposed for solving the variable-order fractional cable equation and a mobile-immobile advection-dispersion equation. In [48], other fractional spectral method is studied for solving linear and nonlinear V-FDEs. Chen et al. [49] used the Legendre wavelets for solving a class of nonlinear V-FDEs and Bhrawy and Zaky [50] have proposed a numerical algorithm for solving V-FDEs with Dirichlet boundary conditions. In [51] an optimization method based on the Legendre wavelets has been proposed for solving variable-order fractional Poisson equation. The authors of [52] proposed an operational matrix method based on the second kind Chebyshev polynomials for solving variable-order fractional biharmonic equation. In [53], the authors proposed a new wavelet method for variable order fractional optimal control problems. Recently, Heydari [54] has proposed a direct method based on the Chebyshev cardinal functions for solving variable-order fractional optimal control problems.

Wavelets theory is an effective subject in mathematics [55] and efficient tool in functional and signal analysis through a basis derived by translations and dilations of a localized wave function [55] namely the mother wavelet. It successfully plays a role in analysis of acoustic and seismic signal processing, signal/image compression and time-frequency of sophisticated signals [55]. Among wavelets, orthogonal wavelets have been widely used for solving various types of constant-order fractional differential equations e.g. [56–68]. The main facility provided by using wavelets is reducing a functional equation to a system of algebraic equations [69]. Moreover, wavelets are useful basis functions with many advantages in computation which are mentioned in [70].

The CWs as a specific kind of orthonormal wavelets have both properties of wavelets (resolution and structure of matrices) and Chebyshev polynomials (orthogonality and spectral accuracy). So, they have been successfully used to solve different types of mathematical systems in recent years e.g. [56–59,69,71]. In [56], a new operational matrix of fractional derivative is built by CWs for solving a system of nonlinear singular fractional Volterra integro-differential equations. The numerical methods based on the CWs are proposed for solving multi-order FDEs in [57], nonlinear fractional integro-differential equations in [58] and generalized Burgers–Huxley equation in [69]. Also, a numerical method through the operational matrices of derivative and integrating the CWs is proposed in [71] for solving PDEs of the telegraph type in conjunction with Dirichlet boundary conditions. In [59], a class of nonlinear FDEs has been solved by using an operational matrix of fractional integration for the CWs.

We remind that an important class of fractional partial differential equations (FPDEs) which has been widely studied in recent years is the time fractional diffusion-wave equation (TFD-WE). This equation is obtained from the classical diffusion-wave equation by replacing the second-order time derivative term by a fractional derivative of order $1<\alpha<2$ [62]. This equation describes important physical phenomena that arise in colloid, amorphous, glassy and porous materials, in fractals and percolation clusters, dielectrics and semiconductors, comb structures, polymers, random and disordered media, biological systems, geophysical and geological processes (see [72] and references therein). Moreover, it should be mentioned that many of the universal electromagnetic, acoustic and mechanical responses can be modeled accurately using the TFD-WE [73,74]. In some practical situations the underlying processes cannot be described by single TFD-WE, but can be modeled using the multi-term time-fractional diffusion-wave equation (M-TFD-WE) form that is given in [72,75] by

$$\mathbf{P}_{\alpha,\alpha_1,\ldots,\alpha_{m_1}}({}_0^cD_t^{\alpha})u(x,t)=\kappa u_{xx}(x,t)+f(x,t),\qquad 0<x<L,\quad t>0, \tag{1.1}$$

where

$$\mathbf{P}_{\alpha,\alpha_1,\ldots,\alpha_{m_1}}({}_0^cD_t^{\alpha})u(x,t)=\left({}_0^cD_t^{\alpha}+\sum_{j=1}^{m_1}d_j\,{}_0^cD_t^{\alpha_j}\right)u(x,t), \tag{1.2}$$

$1<\alpha_{m_1}<\cdots<\alpha_1<\alpha\le 2$, and $d_j\ge 0$ $(j=1,2,\ldots,m_1,\ m_1\in\mathbb{N})$ and $\kappa>0$ are constants. $f(x,t)$ is a sufficiently smooth function and ${}_0^cD_t^{\alpha_j}$ represents the Caputo fractional derivative of order α_j with respect to t.

We remind that the constant-order fractional derivatives are the particular cases of the variable-order ones. In this study, the M-V-TFD-E is generalized by using the concept of V-O fractional derivatives. In other words, the fractional derivative terms in the M-TFD-WE are replaced by V-O fractional derivatives which makes it more competent for modelling but more difficult for solving. Therefore, we propose an efficient and accurate wavelet method for approximating the following M-V-TFD-E:

$$\mathbf{P}_{\alpha(x,t),\alpha_1(x,t),\ldots,\alpha_{m_1}(x,t)}\left({}_0^cD_t^{\alpha(x,t)}\right)u(x,t)+\rho\,u_t(x,t)=\kappa u_{xx}(x,t)+f(x,t),\qquad (x,t)\in[0,1]\times[0,1], \tag{1.3}$$