ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The quenching of solutions to time–space fractional Kawarada problems

Joshua L. Padgett

Texas Tech University, Department of Mathematics and Statistics, Broadway and Boston, Lubbock, TX 79409-1042, United States

ARTICLE INFO

Article history:
Received 26 November 2017
Received in revised form 13 June 2018
Accepted 3 July 2018
Available online xxxx

Keywords:
Kawarada problem
Quenching solution
Caputo derivative
Fractional laplacian
Local existence and uniqueness
Positivity and monotonicity

ABSTRACT

Quenching solutions to a Kawarada problem with a Caputo time-fractional derivative and a fractional Laplacian are considered. The solutions to such problems may only exist locally in time when quenching occurs. Quenching and non-quenching solutions are shown to remain positive and be monotonically increasing in time under minor restrictions. Conditions for quenching to occur are demonstrated and shown to depend on the domain size.

1. Introduction

The purpose of this paper is to investigate the quenching phenomenon of a time–space fractional semilinear equation. Let Ω be an open bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$. We then define $Q_T := \Omega \times (0, T)$ and the parabolic boundary $\Gamma_T = \partial\Omega \times (0, T)$. Consider the following nonlocal Kawarada problem:

$$\begin{cases} \partial_t^\alpha u = -(-\Delta)^s u + f(u), & (x, t) \in Q_t, \\ u = 0, & (x, t) \in \Gamma_T, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where ∂_t^α denotes the Caputo time-fractional derivative of order $\alpha \in (0, 1)$, and $(-\Delta)^s$ is the fractional Laplacian with $s \in (0, 1)$, and the continuous initial data $u_0 : \Omega \to \mathbb{R}^+$ is such that $0 \le u_0 \ll c$. The nonlinear reaction term $f : B_\rho \to \mathbb{R}^+$, where $0 < \rho < c$ and $B_\rho := \{u \in L^\infty(\Omega) : \|u\|_\infty < \rho\}$, is a given continuous, convex function satisfying a local Lipschitz condition on B_ρ. That is, for $u, v \in B_\rho$ there exists a continuous function $L_f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(u) - f(v)\|_{\mathbb{H}^s(\Omega)} \le L_f(c)\|u - v\|_{\mathbb{H}^s(\Omega)}. \tag{1.2}$$

The norm $\|\cdot\|_{\mathbb{H}^s(\Omega)}$ will be defined in the following section. We further assume that f is a monotonically increasing function on B_ρ and

$$\lim_{u \to c^-} f(u) = +\infty. \tag{1.3}$$

E-mail address: joshua.padgett@ttu.edu.

https://doi.org/10.1016/j.camwa.2018.07.009

When $\alpha = s = 1$, (1.1) reduces to the following local semilinear problem

$$\begin{cases} \partial_t u = -(-\Delta)^s u + f(u), & (x,t) \in Q_t, \\ u = 0, & (x,t) \in \Gamma_T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

which was originally studied by Kawarada [1]. This local problem has been well-studied due to the fact that it models several idealized physical phenomena, including solid-fuel combustion and microelectromechanical systems (MEMS) [2–8]. For (1.4), it is known that under certain conditions monotonically increasing solutions to the problem may only exist locally [6,7,9,10]. Further, it is known that for a given function f, the existence of global solutions to (1.4) depends on the spatial domain *size* and *shape* [7,11,12]. We say that two $d-$dimensional spatial domains Ω_1 and Ω_2 have the same shape if there exists $y \in \Omega_1 \cap \Omega_2$ and a constant $\lambda > 0$ such that

$$\Omega_2 = \{z \,:\, z = y + \lambda(x - y), \text{ for } x \in \Omega_1\}. \tag{1.5}$$

Thus, for a *fixed domain shape*, determining whether global solutions to (1.4) exists, reduces to studying the following steady-state problem

$$\begin{cases} \Delta u + \lambda^2 f(u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.6}$$

The existence of a unique positive solution to (1.6) depends on the value of λ, and thus, there is a *critical domain size* that determines whether the classical Kawarada problems emit a global solution [11]. That is, there is a $\lambda_* > 0$ associated to Ω such that if $0 < \lambda < \lambda_*$, then the solution exists globally, and if $\lambda_* < \lambda < \infty$, then there exists a time $T_* < \infty$ such that the maximal interval of existence for the solution is $[0, T_*)$. In this latter case, the solution is said to *quench in finite time*. For the case when $\lambda = \lambda_*$, we say that the solution *quenches in infinite time*, as $T_* = \infty$.

The purpose of this current study is to extend some of the existing results for (1.4) to the nonlocal problem (1.1). We note that this extension is not simply an interesting mathematical problem, but is motivated by numerous physical applications. That is, there have been numerous recent works which outline the importance of fractional and nonlocal models in the accurate modeling of multiphysics problems exhibiting anomalous diffusion [13–16]. In particular, solid-fuel combustion has been shown to behave in a nonlocal manner, thus necessitating the need for considerations of such mathematical models [17].

In order to study the problem (1.1), we introduce the following definition of quenching in the nonlocal setting.

Definition 1.1. A solution u of (1.1) is said to *quench in finite time* if there exists $T_* < \infty$ such that

$$\max\left\{\|u(x,t)\|_\infty \,:\, x \in \overline{\Omega}\right\} \to c^- \text{ as } t \to T_*^-. \tag{1.7}$$

If (1.7) holds for $T_* = \infty$, then u is said to *quench in infinite time*. T_* is referred to as the *quenching time*. The set $\Omega_c \subseteq \Omega$ containing all quenching points is called the *quenching set*.

The paper is organized as follows. In the following section we introduce some important mathematical preliminaries, which are vital to the current study. In Section 3 we consider properties of the operators which generate the solution to (1.1). In Sections 4 and 5 we determine conditions under which there exist unique continuous solutions to (1.1) that are both positive and monotonically increasing on the domain of existence. Section 6 is concerned with establishing conditions under which quenching occurs. Finally, Section 7 provides concluding remarks regarding the current work.

2. Mathematical preliminaries

We now introduce some basic facts and definitions from fractional calculus. In the following, we let $I := (0, T)$ and $\Gamma(\cdot)$ be Euler's gamma function. Further, for $\alpha > 0$ we define the following function

$$g_\alpha(t) = \begin{cases} t^{\alpha-1}/\Gamma(\alpha), & t > 0, \\ 0, & t \le 0, \end{cases} \tag{2.1}$$

with $g_0(t) \equiv 0$.

Definition 2.1. Let $v \in L^1(I)$ and $\alpha \ge 0$. The Riemann–Liouville fractional integral of order α of v is defined as

$$J_t^\alpha v(t) := (g_\alpha * v)(t) = \int_0^t g_\alpha(t-s) u(s)\, ds, \quad t > 0,$$

where $J_t^0 v(t) = v(t)$.

Please cite this article in press as: J.L. Padgett, The quenching of solutions to time–space fractional Kawarada problems, Computers and Mathematics with Applications (2018), https://doi.org/10.1016/j.camwa.2018.07.009.