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Research article

Improved delay-dependent stability result for neural networks with time-varying delays

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ABSTRACT

This paper is concerned with a new Lyapunov-Krasovskii functional (LKF) approach to the stability for neural networks with time-varying delays. The LKF has two features: First, it can make full use of the information of the activation function. Second, it employs the information of the maximal delayed state as well as the instant state and the delayed state. When estimating the derivative of the LKF we employ a new technique that has two characteristics: One is that Wirtinger-based integral inequality and an extended reciprocally convex inequality are jointly employed; the other is that the information of the activation function is used as much as we can. Based on Lyapunov stability theory, a new stability result is obtained. Finally, three examples are given to illustrate the stability result is less conservative than some recently reported ones.

1. Introduction

In the past decades, neural network (NN) has been successfully applied in signal processing, pattern recognition, associative memory, optimization problem, and other engineering and scientific areas [1,2]. However, during the implementation of artificial NNs, the finite switching speed of amplifiers and the inherent communication time between the neurons inevitably introduce time delay, which might cause oscillation, divergence, and even instability. Therefore, the stability of the neural networks with a time-varying delay (DNNs) has attracted a large number of researchers [3], and some stability criteria have been reported in the literature. The stability criteria developed for DNNs can be divided into delay-independent ones and delay-dependent ones. Compared to the former, the delay-dependent stability criteria, which include the information of time delay, usually have less conservative, especially when applied to DNNs with small delay. Thus, more attentions have been paid to delay-dependent stability analysis and its main goal is to reduce the conservatism of the derived stability condition.

For Lyapunov functional approach to delay-dependent stability, the conservatism of the derived stability condition is related to the choosing of the LKF and dealing with its derivative. Constructing a generalized LKF is an effective way to reduce conservatism of the stability results obtained, and various types of LKF have been reported, such as augmented LKF [4], delay-partitioning based LKF [5–8], multiple integrals-based LKF [5,9], activation function based LKF [10], and so on. The technique of dealing with the derivative of the LKF also plays a key role in the process of deriving less conservative stability criteria and numerous techniques have been developed, such as introducing slack variables [11–13], utilizing integral inequality [5,6,14], adopting convex combination technique [15–17], and so on.

Recently stability results for neural networks have been reported in the literature. In Ref. [18], asymptotic stability criterion was obtained using a LKF including a triple integral, where the Wirtinger-based inequality was employed to estimate the derivative of the LKF. In Ref. [19], by defining a more general LKF, a delay-dependent stability result was formulated in linear matrix inequality, while a combined convex approach to stability for DNN was studied in Ref. [20]. Very recently stability analysis was conducted in Ref. [21] and a new stability result was derived, where the tradeoff between conservatism and complexity was considered. In Ref. [22], a delay partitioning method was employed to derive a delay-dependent stability result. The free matrix approach to stability for DNN was studied in Ref. [23]. In Ref. [24], by introducing slack variables some less conservative conditions were obtained; while in Ref. [25] stability criteria were derived by employing a generalized free-weighting-matrix technique. However, all the papers above left room for LKF to improve, with the LKFs not making enough use of the activation function or the time delay. On the other hand, there was some room for those papers to improve in estimating the derivative of the LKF, since new inequalities have already been reported in recently

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published literature. All in all, in terms of the LKF construction or the estimating approach for the derivative of the LKF, each of those papers has its limitations. Hence, there is still room for the stability results in those papers to improve. This motivates the study of this paper.

In this paper, attention is focused on revisiting the stability analysis problem for DNNs. Compared with recently published papers, this paper features:

- A new LKF is constructed with more information of the activation function and the information of the maximal delayed state, the delayed state and the instant state.
- When estimating the derivative of the LKF, Wirtinger-based integral inequality and an extended reciprocally convex inequality are jointly adopted, and more information of the activation function is taken into account.
- A new delay-dependent stability criterion is derived based on Lyapunov stability theory, and the delay-dependent stability criterion has less conservatism.

Notations: Throughout this paper $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices; the superscript 'T' and '-1' stand for the transpose and inverse of a matrix, respectively; 'I' and '0' represent the identity and null matrices with appropriate dimensions, respectively; the notation $|\cdot|$ denotes the absolute value; the notation diag$\{\cdots\}$ stands for a block-diagonal matrix; the notation $P > 0(\geq 0)$ means that P is a real symmetric and positive-definite (semipositive-definite) matrix. Moreover, for any square matrix A, we define $Sym\{A\} = A + A^T$, and the symmetric term in the matrix is denoted by $*$.

2. Problem formulation

Consider the generalized DNN with a time-varying delay $\tau(t)$ [21,26]:

$$\dot{u}(t) = -Au(t) + W_0 g(Wu(t)) + W_1 g(Wu(t-\tau(t))) + \varrho \tag{1}$$

where $u(t) = [u_1(t)\ \ u_2\ (t)\ \cdots\ u_n\ (t)]^T$ is the state vector associated with the n neurons; $g(\cdot) = [g_1(\cdot)\ \ g_2\ (\cdot)\ \cdots\ g_n(\cdot)]^T$ represents the neuron activation function with $g(0) = 0$; $A = \text{diag}\{a_1, a_2, \cdots, a_n\} > 0$; W, W_0 and W_1 are the connection weight matrices; $\varrho = [\varrho_1\ \ \varrho_2\ \ \cdots\ \ \varrho_n]^T$ is a vector representing the bias; and $\tau(t)$ is a time-varying delay satisfying

$$0 \leq \tau(t) \leq h,\ \dot{\tau}(t) \leq \mu \tag{2}$$

The following assumption is made throughout this paper.

Assumption 1. The neuron activation function is bounded, and satisfies

$$l_i^- \leq \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \leq l_i^+,\ s_1 \neq s_2,\ i = 1, 2, \cdots, n$$

where l_i^- and l_i^+ are known real constants.

Based on Assumption 1, there exists an equilibrium point u^* for (1), i.e.

$$0 = -Au^* + W_0 g(Wu^*) + W_1 g(Wu^*) + \varrho$$

To transfer the equilibrium u^* to the origin, we make the transformation $x(t) = u(t) - u^*$ to neural network (1). Then, it becomes

$$\dot{x}(t) = -Ax(t) + W_0 f(Wx(t)) + W_1 f(Wx(t-\tau(t))) \tag{3}$$

where $x(t) = [x_1(t)\ \ x_2\ (t)\ \cdots\ x_n\ (t)]^T$ is the state vector of the transformed system (3), $f(\cdot) = [f_1(\cdot)\ \ f_2\ (\cdot)\ \cdots\ f_n\ (\cdot)]^T$ and $f_i(w_i x(t)) = g_i(w_i x(t) + w_i u^*) - g_i(w_i u^*)$ with $f_i(0) = 0$ and w_i denoting the ith row vector of the matrix W. It is noted that

$$l_i^- \leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq l_i^+,\ s_1 \neq s_2,\ i = 1, 2, \cdots, n \tag{4}$$

which implies

$$l_i^- \leq \frac{f_i(s)}{s} \leq l_i^+,\ s \neq 0,\ i = 1, 2, \cdots, n$$

This paper aims to derive a delay-dependent stability criterion of DNN (3) with (2) and (4) to determine the admissible upper bound of $\tau(t)$, which can guarantee the stability of the DNN. To the end, we need the following lemmas:

Lemma 1. **([27] Jensen's Inequality).** *For any matrix $0 < R \in \mathbb{R}^{n\times n}$, scalars $\alpha < \beta$, and vector $x: [\alpha, \beta] \to \mathbb{R}^n$, such that the integration concerned is well defined, then*

$$(\beta - \alpha)\int_\alpha^\beta x^T(s)Rx(s)ds \geq \int_\alpha^\beta x^T(s)dsR\int_\alpha^\beta x(s)ds$$

Lemma 2. ([28]). *For any matrix $0 < R \in \mathbb{R}^{n\times n}$, scalars $\beta > \alpha \geq 0$ and vector $x: [\alpha, \beta] \to \mathbb{R}^n$, such that the integration concerned is well defined, then*

$$\begin{aligned}&\int_\alpha^\beta (s-\alpha)x^T(s)Rx(s)ds\\ \geq\ & \tfrac{2}{(\beta-\alpha)^2}\int_\alpha^\beta (s-\alpha)x^T(s)dsR\int_\alpha^\beta (s-\alpha)x(s)ds\end{aligned}$$

Lemma 3. **([29] Wirtinger-based integral inequality).** *For any matrix $0 < R \in \mathbb{R}^{n\times n}$, and any differentiable function $x: [\alpha, \beta] \to \mathbb{R}^n$, the following inequality holds:*

$$\int_\alpha^\beta \dot{x}^T(s)R\dot{x}(s)ds \geq \frac{1}{\beta - \alpha}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}^T\begin{bmatrix}R & 0\\ 0 & 3R\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}$$

where

$$\begin{aligned}\varphi_1 &= x(\beta) - x(\alpha)\\ \varphi_2 &= x(\beta) + x(\alpha) - \tfrac{2}{\beta-\alpha}\int_\alpha^\beta x(s)ds.\end{aligned}$$

Lemma 4. **([30] Extended reciprocally convex inequality).** *For any matrix $0 < R_i \in \mathbb{R}^{n\times n}(i = 1, 2)$, if there exist symmetric matrices X_1, $X_2 \in \mathbb{R}^{n\times n}$ and any matrices $Y_1, Y_2 \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix}R_1 - X_1 & -Y_1\\ * & R_2\end{bmatrix} > 0,\ \begin{bmatrix}R_1 & -Y_2\\ * & R_2 - X_2\end{bmatrix} > 0$$

then the following inequality holds for all $\alpha \in [0, 1]$

$$\begin{bmatrix}\frac{1}{\alpha}R_1 & 0\\ * & \frac{1}{1-\alpha}R_2\end{bmatrix} \geq \begin{bmatrix}R_1 & 0\\ * & R_2\end{bmatrix} + (1-\alpha)\begin{bmatrix}X_1 & Y_2\\ * & 0\end{bmatrix} + \alpha\begin{bmatrix}0 & Y_1\\ * & X_2\end{bmatrix}$$

Remark 1. Let $X_1 = X_2 = 0$, $Y_1 = Y_2 = Y$, and then Lemma 4 reduces to the reciprocally convex inequality lemma [31]. So, the Lemma 4 is refered to as an extended reciprocally convex inequality.

3. Main result

In this section, we will construct a new LKF to derive a delay-dependent stability result for DNN (3) with (2) and (4). To simplify the representation, we introduce some notations as follows: