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Maximal Covering Location Problems on networks with regional demand[☆,☆☆]

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ABSTRACT

Covering problems are well studied in the Operations Research literature under the assumption that both the set of users and the set of potential facilities are finite. In this paper, we address the following variant, which leads to a Mixed Integer Nonlinear Program (MINLP): locations of p facilities are sought along the edges of a network so that the expected demand covered is maximized, where demand is continuously distributed along the edges. This MINLP has a combinatorial part (which edges of the network are chosen to contain facilities) and a continuous global optimization part (once the edges are chosen, which are the optimal locations within such edges).

A branch-and-bound algorithm is proposed, which exploits the structure of the problem: specialized data structures are introduced to successfully cope with the combinatorial part, inserted in a geometric branch-and-bound algorithm.

Computational results are presented, showing the appropriateness of our procedure to solve covering problems for small (but non-trivial) values of p.

1. Introduction

The Maximal Covering Location Problem, (MCLP), [3,14,15,22], is a classic problem in locational analysis with applications in a good number of fields, such as health care, emergency planning, ecology, statistical classification, homeland security, see e.g. [1,8,13,18,39,40] and the references therein. Given a finite set of users A, each $a \in A$ with demand $\omega_a \geq 0$, a set of p facilities in a set F is sought in order to maximize the demand covered. A point is said to be covered by a set $F^* \subset F$ of p facilities if there is at least one $f \in F^*$ at distance from a not greater than R, where $R > 0$ is a fixed number, called the *covering radius*.

(MCLP) is easily expressed as an Integer Program. Indeed, defining binary variables y_f and z_a to indicate respectively whether a facility at f is open, and whether a is covered, (MCLP) amounts to solving the following program:

$$\begin{aligned} \max \quad & \sum_{a \in A} \omega_a z_a \\ \text{s.t.} \quad & z_a \leq \sum_{f \in F:\, d(a,f) \leq R} y_f \quad \forall a \in A \\ & \sum_{f \in F} y_f = p \\ & y_f \in \{0,1\} \quad \forall f \in F \\ & z_a \in \{0,1\} \quad \forall a \in A. \end{aligned} \tag{1}$$

(MCLP) is known to be NP-hard, [27], but formulated as (1) is, in words of [37], integer-friendly, in the sense that its continuous relaxation is often all-integer, and thus no much branching is usually needed in a branch-and-bound algorithm. See [23,29,36,38] and the references therein for heuristic approaches to handle problems of larger size.

Extensions and closely related models to the (MCLP) abound in the Operations Research literature. First, (MCLP) has been studied assuming that the space is not a discrete set but a network: the set A of users is the set of nodes of a network N, and facilities are allowed to be located not only at the nodes, but anywhere on N. It is shown, however, that one only needs to consider a finite and relatively small set of candidate locations, [14,27], and thus the problem can be written in the form of (MCLP) above. Nontrivial

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extensions include, for instance, replacing the basic yes/no covering function to more general decreasing functions in the distance separating the user and the facility, [3,4,2,5]; another variant is found when the set A of users is finite, but the feasible locations are assumed to be a subset of the plane, yielding planar covering models, as reviewed in [33].

Much less literature exists on covering models with *regional demand*, [21,26,31], in which, by the very nature of the problem, assuming the demand to be concentrated at a finite set (e.g. centroids of neighbourhoods, towns, administrative units or census boundaries, [31]) is a crude approximation. The consequences of inaccuracies due to such discretization are well studied, [16,28,31], and thus demand is advocated to be modeled as following a continuous distribution on a given region. See also [9–11] for other location models with continuously distributed demand.

The following version of the classic (MCLP) with regional demand is addressed in this paper: demand is assumed to be continuously distributed along the edges of a network and p points along the set of edges of the network are sought in order to maximize the expected covering of the demand. Hence, the model differs from the classic (MCLP) in two main issues: first, the set of feasible locations is not a discrete set, but (a set of) the edges of a network; moreover, demand is assumed here to be distributed along the edges of the network, making it a realistic model, for instance, for covering problems in an urban context, in which users are located along streets (the edges), or for the location of emergency services to attend accidents, which take place along the roads (edges of the transportation network).

Let us now introduce formally the problem under consideration. We are given a network $N=(V,E)$; each edge $e \in E$ has associated its length l_e, which allows us to talk about points in an edge: edge e, with endpoints u, v, is identified with the interval $[0, l_e]$, and we thus identify any $x \in [0, l_e]$ as the point in the edge e at distance x of u and distance $l_e - x$ of v. With this identification, the shortest-path distance between the nodes in V is readily extended to a metric d on the points in the edges. Moreover, each edge e has a weight $\omega_e \geq 0$ and a probability density function (pdf) f_e, which models the demand along edge e. We assume that a radius $R>0$ is given, and a point x along an edge $e \in E$ is covered by the set of facilities at $t_1, \dots, t_p$ if

$$\min_{1 \leq i \leq p} d(t_i, x) \leq R. \tag{2}$$

The expected demand of edge e covered by facilities at $\mathbf{t}=(t_1, \dots, t_p)$ is given by

$$\omega_e \int_0^{l_e} \delta_e(x;\mathbf{t}) f_e(x)\, dx,$$

where $\delta_e(x;\mathbf{t})$ takes the value 1 when $x \in e$ is covered by facilities at $\mathbf{t}=(t_1, \dots, t_p)$, i.e., when (2) is fulfilled, and takes the value 0 otherwise.

With this, the optimization problem at hand can be written as

$$\max_{\mathbf{t} \in E^p} C(\mathbf{t}) := \sum_{e \in E} \omega_e \int_0^{l_e} \delta_e(x;\mathbf{t}) f_e(x)\, dx. \tag{3}$$

The remainder of this note is structured as follows. In Section 2, structural properties of the MINLP (3) are studied. A branch-and-bound method is designed in Section 3. Exploiting the structure of the problem, data structures and bounding procedures are proposed, and they are tested on a set of instances in Section 4. The paper ends with some concluding remarks and possible extensions in Section 5.

2. Structural properties

Property 2.1. *For any p-tuple of edges $(e_1, \dots, e_p) \in E^p$, the function $C: \mathbf{t}=(t_1, \dots, t_p) \in [0, l_{e_1}] \times \dots \times [0, l_{e_p}] \longrightarrow C(\mathbf{t})$ is continuous in $[0, l_{e_1}] \times \dots \times [0, l_{e_p}]$.*

Proof. Using the inclusion-exclusion principle, we can re-write $C(\mathbf{t})$ as

$$C(\mathbf{t}) = \sum_{e \in E} \omega_e \int_0^{l_e} \sum_{I \subset \{1,\dots,p\}} (-1)^{1+|I|} \prod_{i \in I} \delta_e(x;t_i) f_e(x)\, dx.$$

Hence, it suffices to show that, for any $e=(u,v) \in E$ and any nonempty I, the function $\int_0^{l_e} \prod_{i \in I} \delta_e(x;t_i) f_e(x)\, dx$ is continuous in $\mathbf{t}$. Split the index set I in those indices corresponding to facilities in e and not in e respectively:

$$I_+ := \{i \in I : e_i = e\}$$
$$I_- := \{i \in I : e_i \neq e\}.$$

Observe that, for $i \in I_+$, one has

$$\delta_e(x;t_i) = 1 \quad \text{iff} \quad d(x,t_i) \leq R \quad \text{iff} \quad x \in [t_i - R, t_i + R],$$

while for $i \in I_-$,

$$\delta_e(x;t_i) = 1 \quad \text{iff} \quad \min\{x + d(u,t_i), l_e - x + d(v,t_i)\} \leq R \quad \text{iff} \quad x \in [0, R - d(u,t_i)] \cup [d(v,t_i) + l_e - R, l_e]$$

Hence

$$\prod_{i \in I_+} \delta_e(x;t_i) = 1 \quad \text{iff} \quad x \in \left[\max_{i \in I_+} t_i - R, \min_{i \in I_+} t_i + R\right]$$

$$\prod i \in I_- \delta_e[x;t_i] = 1 \quad \text{iff} \quad x \in \left[0, R - \max_{i \in I_-} d(u,t_i)\right] \cup \left[\max_{i \in I_-} d(v,t_i) + l_e - R, l_e\right] \prod_{i \in I} \delta_e(x;t_i) = 1 \quad \text{iff}$$

$$x \in \left[\max\left\{\max_{i \in I_+} t_i - R, 0\right\}, \min\left\{\min_{i \in I_+} t_i + R, R - \max_{i \in I_-} d(u,t_i)\right\}\right]$$
$$\cup \left[\max\left\{\max_{i \in I_+} t_i - R, \max_{i \in I_-} d(v,t_i) + l_e - R\right\}, \min\left\{\min_{i \in I_+} t_i + R, l_e\right\}\right]$$
$$= [a_1(\mathbf{t}), b_1(\mathbf{t})] \cup [a_2(\mathbf{t}), b_2(\mathbf{t})].$$

Hence,

$$\int_0^{l_e} \prod_{i \in I} \delta_e(x;t_i) f_e(x)\, dx = \int_{[a_1(\mathbf{t}),b_1(\mathbf{t})] \cup [a_2(\mathbf{t}),b_2(\mathbf{t})]} f_e(x)\, dx$$
$$= \int_{a_1(\mathbf{t})}^{b_1(\mathbf{t})} f_e(x)\, dx + \int_{a_2(\mathbf{t})}^{b_2(\mathbf{t})} f_e(x)\, dx - \int_{\max\{a_1(\mathbf{t}),a_2(\mathbf{t})\}}^{\min\{b_1(\mathbf{t}),b_2(\mathbf{t})\}} f_e(x) dx$$
$$= \max\{F_e(b_1(\mathbf{t})) - F_e(a_1(\mathbf{t})), 0\} + \max\{F_e(b_2(\mathbf{t})) - F_e(a_2(\mathbf{t})), 0\}$$
$$- \max\{F_e(\min\{b_1(\mathbf{t}), b_2(\mathbf{t})\}) - F_e(\max\{a_1(\mathbf{t}), a_2(\mathbf{t})\}), 0\},$$

where F_e is the cumulative distribution function associated with the pdf f_e. Since F_e is continuous, $C(\mathbf{t})$ is continuous as well. □

Once the p-tuple of edges $(e_1, \dots, e_p)$ is chosen, the function C is continuous on the compact set $[0, l_{e_1}] \times \dots \times [0, l_{e_p}]$, and attains its maximum. Since the possible choices of p-tuple of edges is also finite, the maximum of C on E^p is attained. Finding such maximum may be hard because, for arbitrary pdfs f_e defining the demand along the edges, the function C may not be convex, and thus Global Optimization techniques are to be used; in its full generality, C may lack important structural properties, such as Lipschitz-continuity. This is shown in the following example.

Example 2.1. Consider a graph $N=(V,E)$ with two nodes, v_1, v_2, connected by an edge e of length 2, so that we can identify the edge with the segment $[-1,1]$ and the nodes with the endpoints