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Strict Majority Bootstrap Percolation in the r-wheel ☆

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ABSTRACT

In the strict Majority Bootstrap Percolation process each passive vertex v becomes active if at least $\lceil\frac{\deg(v)+1}{2}\rceil$ of its neighbors are active (and thereafter never changes its state). We address the problem of finding graphs for which a small proportion of initial active vertices is likely to eventually make all vertices active. We study the problem on a ring of n vertices augmented with a "central" vertex u. Each vertex in the ring, besides being connected to u, is connected to its r closest neighbors to the left and to the right. We prove that if vertices are initially active with probability $p > 1/4$ then, for large values of r, percolation occurs with probability arbitrarily close to 1 as $n \to \infty$. Also, if $p < 1/4$, then the probability of percolation is bounded away from 1.

1. Introduction

Consider the following deterministic process on a graph $G = (V, E)$. Initially, every vertex in V can be either *active* or *passive*. A passive vertex v becomes active iff at least k of its neighbors are already active; once active, a vertex never changes its state. This process is known as *k-neighbor bootstrap percolation* [4]. If at the end of the process all vertices are active, then we say that the initial set of active vertices *percolates*. We wish to determine the minimum ratio of initially active vertices needed to achieve percolation with high probability. More precisely, suppose that the elements of the initial set of active vertices $A \subseteq V$ are chosen independently with probability p. The problem is finding the least p for which percolation of A is likely to occur.

Since its introduction by Chalupa et al. [4], the bootstrap percolation process has mainly been studied in the d-dimensional grid $[n]^d = \{1, \dots, n\}^d$ [1]. The precise definition of *critical probability* that has been used is the following:

$$p_c([n]^d, k) = \inf\{p \in [0,1]: \mathbb{P}_p(A \text{ percolates } [n]^d) \geqslant 1/2\}.$$

In [1] it is proved that, for every $d \geqslant k \geqslant 2$, $p_c([n]^d, k) = (\frac{\lambda(d,k)+o(1)}{\log_{(k-1)} n})^{d-k+1}$, where $\lambda(d,k) < \infty$ are equal to the values of specific definite integrals for every $d \geqslant k \geqslant 2$. In the (*simple*) *Majority Bootstrap Percolation* (*simple MBP*) process (introduced in [2]) each passive vertex v becomes active iff at least $\lceil\frac{\deg(v)}{2}\rceil$ of its neighbors are active, where $\deg(v)$ denotes the degree of vertex v in G. Note that for $[n]^d$, the critical probability for simple MBP percolation corresponds to $p_c([n]^d, d)$, which goes to 0 as $n \to \infty$.

Here we introduce the *strict Majority Bootstrap Percolation* (*strict MBP*) process: each passive vertex v becomes active iff at least $\lceil\frac{\deg(v)+1}{2}\rceil$ of its neighbors are active. Note that if $\deg(v)$ is odd, then strict and simple MBP coincide. For $[n]^d$ the critical probability in strict MBP $p_c([n]^d, d+1)$ goes to 1. This holds because, in this case, any unit hypercube starting with its 2^d corners passive will stay passive forever.

A natural problem is finding graphs for which the critical probability in the strict MBP is small. Results by Balogh

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and Pittel [3] imply that the critical probability of the strict MBP for random 7-regular graphs is 0.269. In [6], two families of graphs for which the critical probability is also small (but higher than 0.269) are explored. The idea behind these constructions is the following. Consider a regular graph of even degree G. Let $G * u$ denote the graph G augmented with a single universal vertex u. The strict MBP dynamics on $G * u$ has two phases. In the *first phase*, assuming that vertex u is not initially active, the dynamics restricted to G corresponds to the strict MBP. If more than half of the vertices of G become active, then the universal vertex u also becomes active, and the *second phase* begins. In this new phase, the dynamics restricted to G follows the simple MBP (and full activation becomes much more likely to occur).

The two augmented graphs studied in [6] were the *wheel* $\mathrm{WH}_n = u * R_n$ and the *toroidal grid plus a universal vertex* $\mathrm{TWH}_n = u * R_n^2$ (where R_n is the ring on n vertices and R_n^2 is the toroidal grid on n^2 vertices). For a family of graphs $\mathcal{G} = (G_n)_n$, the following parameter was defined (as before, A denotes the initial set of active vertices):

$$p_c^+(\mathcal{G}) = \inf\Big\{p \in [0,1]: \liminf_{n\to\infty} \mathbb{P}_p(A \text{ percolates } G_n \text{ in strict MBP }) = 1\Big\}.$$

Consider the families $\mathcal{WH} = (\mathrm{WH}_n)_n$ and $\mathcal{TWH} = (\mathrm{TWH}_n)_n$. It was proved in [6] that $p_c^+(\mathcal{WH}) = 0.4030\ldots$. For the toroidal case it was shown that $0.35 \leqslant p_c^+(\mathcal{TWH}) \leqslant 0.372$. Computing the critical probability for the wheel is trivial. Nevertheless, if we increase the *radius* of the vertices, then the situation becomes much more complicated. More precisely, let $R_n(r)$ be the ring where every vertex is connected to its r closest vertices to the left and to its r closest vertices to the right. Here we study the strict MBP process in a generalization of the wheel that we call r*-wheel* $\mathrm{WH}_n(r) = u * R_n(r)$. Our main result is the following:

Theorem 1. *The limit of $p_c^+(\mathcal{WH}(r))$, as $r \to \infty$, exists and equals $1/4$.*

2. Preliminary results

We start by showing that we can reduce our problem to the issue of whether a single fixed (non-universal) vertex eventually becomes active.

Lemma 2. *Let $0 < p < 1$ be the probability for a vertex to be initially active. Let r be a positive integer. Denote by $p_W(n,r,p)$ the percolation probability of the* r-wheel *and denote by $p_R(n,r,p)$ the probability that the strict majority on $R_n(r)$ ends up with (strictly) more active than passive vertices. Then,*

$$\liminf_{n\to\infty} p_R(n,r,p) \leqslant \liminf_{n\to\infty} p_W(n,r,p),$$
$$\limsup_{n\to\infty} p_W(n,r,p) \leqslant p + (1-p)\cdot \limsup_{n\to\infty} p_R(n,r,p).$$

Proof. Note that for $\epsilon > 0$ we can choose n large enough so that the probability that at least one block of r consecutive vertices is initially active is larger than $1-\epsilon$, in which case percolation occurs iff the universal vertex becomes active during the evolution. We deduce the first inequality by taking ϵ arbitrarily small. Note now that the universal vertex is active when the dynamics stabilizes only if it was either already active initially (probability p) or if it was initially passive and the dynamics on the ring $R_n(r)$ produces more than $n/2$ active vertices. $\square$

The vertices of the ring R_n will be denoted as $0, 1, \ldots, n-1$, starting at some arbitrary vertex (arithmetic over vertex indices will always be modulo n). The positive integer r will be called the *radius*.

Lemma 2 shows that we can study the ring $R_n(r)$ and its dynamics to derive results about the r-wheel. Now, consider the 0–1 random variable $X_i(n,r)$ giving the state of vertex i after stabilization of the dynamics ($X_i(n,r) = 0$ if the state is passive, and $X_i(n,r) = 1$ if it is active). Next, we show how to bound $p_R(n,r,p)$ in terms of $\mathbb{E}_p(X_0(n,r))$.

Lemma 3. *Let $0 < p < 1$, $n \in \mathbb{N}^+$, and r be a fixed radius. Then,*

$$2\mathbb{E}_p\big(X_0(n,r)\big) - 1 \leqslant p_R(n,r,p) \leqslant 2\mathbb{E}_p\big(X_0(n,r)\big).$$

Proof. By definition $p_R(n,r,p) = \mathbb{P}_p(\sum_i X_i(n,r) > n/2)$. By Markov's inequality we then have $\mathbb{P}_p(\sum_i X_i(n,r) > n/2) \leqslant \frac{2}{n}\mathbb{E}_p(\sum_i X_i(n,r))$. Using linearity of expectation and the fact that all $X_i(n,r)$ are equally distributed (symmetry of the ring), we deduce $p_R(n,r,p) \leqslant 2\mathbb{E}_p(X_0(n,r))$. The lower bound is obtained in the same way considering again Markov's inequality but for the (again positive) random variable $n - \sum_i X_i(n,r)$. More precisely:

$$p_R(n,r,p) = 1 - \mathbb{P}_p\Big(n - \sum_i X_i(n,r) > n/2\Big) \geqslant 1 - \frac{2}{n}\mathbb{E}_p\Big(n - \sum_i X_i(n,r)\Big). \qquad \square$$

3. Lower bound on $p_c^+(\mathcal{WH}(r))$

We will assume $n > 2r+1$ and that the initial state of the universal vertex u is passive. Let $0 < p < 1/2$ and $q = 1-p$. The starting configuration $\sigma = (\sigma_0, \ldots, \sigma_{n-1})$, where vertex j is initially active (respectively passive) if and only if $\sigma_j = 1$ (respectively $\sigma_j = 0$), occurs with probability $p^{\sum_j \sigma_j} q^{n - \sum_j \sigma_j}$. We write X_0 instead of $X_0(n,r)$. Conditioning on σ_0,

$$\mathbb{P}_p(X_0 = 1) \leqslant p + \mathbb{P}_p(X_0 = 1 | \sigma_0 = 0). \tag{1}$$

We say there is a *wall* located $\ell > 0$ vertices to the left of vertex 0 if $\sigma_{-\ell} = 1$, $\sigma_{-\ell-1} = \sigma_{-\ell-2} = \cdots = \sigma_{-\ell-(r+1)} = 0$. Similarly, we say there is a wall located at $\ell > 0$ vertices to the right of vertex 0 if $\sigma_\ell = 1$, $\sigma_{\ell+1} = \sigma_{\ell+2} = \cdots = \sigma_{\ell+(r+1)} = 0$. Let L (respectively R) be the smallest positive ℓ such that there is a wall located ℓ vertices to the