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Packing unequal circles using formulation space search

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ABSTRACT

In this paper we present a heuristic algorithm for the problem of packing unequal circles in a fixed size container such as the unit circle, the unit square or a rectangle. We view the problem as being one of scaling the radii of the unequal circles so that they can all be packed into the container. Our algorithm is composed of an optimisation phase and an improvement phase. The optimisation phase is based on the formulation space search method whilst the improvement phase creates a perturbation of the current solution by swapping two circles. The instances considered in this work can be categorised into two: instances with large variations in radii and instances with small variations in radii. We consider six different containers: circle, square, rectangle, right-angled isosceles triangle, semicircle and circular quadrant. Computational results show improvements over previous work in the literature.

1. Introduction

The circle packing problem is a well studied problem [40] whose aim is the packing of a certain number of circles, each one with a fixed radius (not necessarily the same for each circle) inside a container. The shape of the container may vary from a circle, a square, a rectangle, etc. It has been applied in different areas of science, amongst some of its applications we can find it in the motor cycle industry [8], circular cutting, communication networks, facility location and dashboard layout [7].

In this paper we will address the problem of packing unequal circles inside six different containers: unit circle, unit square, rectangle of length L and width W, right-angled isosceles triangle with equal sides of length L, unit semicircle and unit circular quadrant. With regard to the variations in the size of the n circles this will be determined by their radii (R_i for all $i=1,\dots,n$). We consider an instance a large variation instance if the radii are defined as $R_i=i$ for all $i=1,\dots,n$ and an instance a small variation instance when their radii are defined as $R_i=1/\sqrt{i}$ for all $i=1,\dots,n$. Generically a *large variation* instance means that there is a wide disparity between the size of the circles, a *small variation* instance means that there is only a small disparity between the sizes of the circles. In Fig. 1(a) and (b) are two examples of unequal circles inside a circular container where the difference between large and small variation can be appreciated if we compare the size of the largest circle with the size of the smallest circle in Fig. 1(a) as compared to Fig. 1(b). A similar situation happens with Fig. 1(c) and (d) where the container is a square.

When the container is a rectangle there are (as discussed in the literature survey section below) differing objectives that have been adopted in the literature (e.g. minimise the perimeter of the container, minimise the area of the container). These objectives regard the circles as being of fixed size and the container as being of variable size. In this paper we reverse this perspective. Namely we view the problem as being one of scaling the radii of the unequal circles (so the circles are now of variable size) so that they can all be packed into the fixed sized container. As far as we are aware this scaled view of the problem has not been considered before in the literature.

The algorithm presented here is a heuristic that consist of two phases, optimisation and improvement: for the optimisation phase we use the formulation space search method with a mixed Cartesian/Polar formulation of the problem, for the improvement phase we use a swapping process aiming to improve the solution obtained in the optimisation phase.

This paper is organised in seven sections as follows. In Section 2 we present the literature review. In Section 3 we present the formulation of the problem in the case of the unit circle container whilst Section 4 describes the detailed algorithm proposed. In Section 5 we present details of the modifications needed when other containers are considered. In Section 6 we present the computational results and the comparison against previously best known solutions (where possible) presenting as well the results for all the containers considered and finally in Section 7 we present conclusions.

2. Literature survey

2.1. Circle packing

The circle packing problem has been subject of study by a wide spectrum of different approaches in the literature both for the

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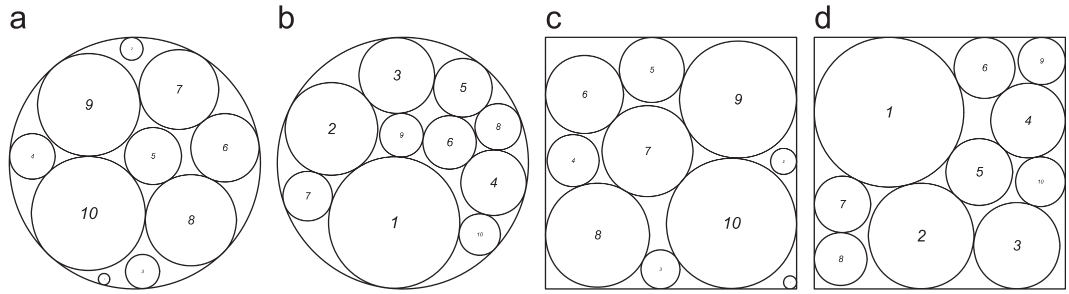

Fig. 1. Circle packing problem with $n=10$ unequal circles. (a) Circle, $R_i = i$. (b) Circle, $R_i = 1/\sqrt{i}$. (c) Square, $R_i = i$. (d) Square, $R_i = 1/\sqrt{i}$.

equal circle case ($R_i = R_j$ $\forall i,j$) and the unequal circle case. Here we will focus on papers that are relevant to our subject: unequal circles and formulation space search. To structure our literature survey we consider circular and rectangular containers separately below. Note here that a review of packing problems has been given in Castillo et al. [7].

2.1.1. Circular container

Some early approaches to solve the problem of packing unequal circles based on a quasi-physical method are presented in [25,26]. Wang et al. [42] incorporate a quasi-physical and a quasi-human approach; the algorithm presented alternates from the quasi-physical to the quasi-human strategy until a solution is found, or the time allowed has been reached. The physical strategy is based on the laws of motion and is used to obtain a packing of n circles until a local minimum is found. The human strategy aims to improve the current solution by picking up circles with maximum pain (depth of overlap) and randomly allocate them inside the container. An improved version is presented by Huang and Chen [22] by changing the condition that allows switching from the quasi-physical to the quasi-human strategy in a shorter period of time; instead of considering a local minimum solution as the condition to switch they change by considering a promising local minimum solution. They give numerical results for four instances where the time required was reduced significantly.

Zhang and Deng [43] combine simulated annealing with tabu search. They give computational results for nine instances, five instances involving up to 91 equally sized circles, four instances involving up to 17 unequally sized circles.

In 2006 a contest was held for finding the optimal solutions for the packing problem for unequal circles with radii $R_i = i$ for all $i = 1, \ldots, n$ inside a circular container of minimum radius. The instances considered were from $n=5$ to $n=50$. In Addis et al. [1] they present the algorithm that made them the winners of that contest. Their approach is based on population basin hopping [12] and reduction of the space of variables. Although they won the contest further computational results are frequently reported. The website [38] maintains the best solutions for different variants of the packing problem and is continually updated.

Huang et al. [24] present two heuristics to solve the problem of packing unequal circles aiming to minimise the radius of the circular container. Their first heuristic is based on the maximum hole degree strategy and their second heuristic uses a self look-ahead strategy to improve the solution obtained by the first heuristic.

Akeb and Li [4] extended the work in [24] by combining two strategies: maximum hole degree and minimum damage. Both strategies are used to determine the position of the next circle to be packed. Depending on the current solution the circle will be located according to one or the other strategy. Akeb et al. [3] propose an algorithm that embeds beam search within a dichotomous local search for the minimum container radius. The beam search branches out a node based on the maximum hole degree measure.

In Hifi and M'Hallah [18] they present a heuristic that dynamically updates the centre and the radius of the circular container after the best position for the current circle to be packed has been found. Afterwards, Hifi and M'Hallah [19] propose an algorithm that combines adaptive and restarting techniques. The algorithm consist of three phases: a dynamic search that determines a packing solution, an adaptive phase that uses intensification and diversification on the solution obtained in the dynamic search (the intensification aims to find a smaller radius for the container whilst the diversification uses different packing techniques for the current solution) and finally a restarting phase that is based on the hill climbing approach.

In Lu and Huang [35] they propose an algorithm to solve the circle packing problem combining two strategies: Prune-Enriched-Rusenbluth Method (PERM) with the maximum cave degree approach [24]. The circles are packed one by one as close as possible according to the maximal cave degree approach forming a partial configuration. The PERM strategy is used in every partial packing to decide which branches to prune, or to enrich, based on the associated estimated weight that every partial configuration has. They present results for 14 instances where the radii of the circles to be packed are equal and unequal.

In Grosso et al. [11] they propose an algorithm based on monotonic basin hopping [29,41] along with its variant, population basin hopping [12], to solve the circle packing problem of equal and unequal circles inside a circular container with minimum radius. Monotonic basin hopping is a single solution iterative local search method with a perturbation move that allows diversification of the solution by jumping from one local minimum to a closer one. Population basin hopping maintains a set of candidate solutions, each member of that set is compared to another one by a dissimilarity measure keeping the one with minimum value function. Computational results for the case of equal circles with up to $n=100$ circles are given, whilst for the case of unequal circles they consider nine instances where the largest instance consists of 162 unequal circles.

Liu et al. [31] use an approach based on energy landscaping paving [14]. This is an iterative approach whereby the centre of a chosen circle is repositioned at each iteration. They give computational results for their approach for the problem of packing up to 100 identical circles of unit radius into a containing circle, as well as for the problem of packing up to 162 unequal circles into a containing circle. Liu et al. [32] extend Liu et al. [31] by introducing gradient descent into the approach. They give computational results for their approach for the problem of packing up to 17 unequal circles into a containing circle. Liu et al. [30] make a number of modifications to the algorithm of Liu et al. [32].

In Al-Mudahka et al. [5] they propose an algorithm that combines two procedures: local search and a nested partition of the feasible space. The local search procedure used is tabu search,