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Solving discretely constrained, mixed linear complementarity problems with applications in energy

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ABSTRACT

This paper presents an approach to solving discretely constrained, mixed linear complementarity problems (DC-MLCPs). Such formulations include a variety of interesting and realistic models of which two are highlighted: a market-clearing auction typical in electric power markets but suitable in other more general contexts, and a network equilibrium suitable to energy markets as well as other grid-based industries. A mixed-integer, linear program is used to solve the DC-MLCP in which both complementarity as well as integrality are allowed to be relaxed. Theoretical and numerical results are provided to validate the approach.

1. Introduction

In this paper, we present a new approach to solve discretely constrained, mixed linear complementarity problems (DC-MLCPs) in which some of the variables are constrained to be integer-valued and some can take on continuous values. This is an important extension of the general MLCP in which all variables are assumed to be continuous and relates for example to Nash–Cournot games in which some of the players' variables are discrete and some are continuous. A mixed-integer linear program (MILP) is presented which solves DC-MLCPs with complementarity and integrality suitably relaxed. As an example in Section 2 shows, enforcing exact complementarity and exact integrality may not be feasible. From a compromise perspective, the MILP that relaxes both of these conditions is somewhat related to the notion of bounded rationality in equilibrium systems as discussed in [22].

This focus on integer variables (and/or related techniques) and one [8] or two-level equilibria, e.g., mathematical programs with equilibrium constraints [21] has seen some research efforts over the years in both modeling and methods (e.g., [5,20,1,23,24,15,13,3]) and joins two important fields of operations research. This work also has relevance to both energy market modeling [25] and network optimization [17].

Section 2 presents a general formulation for an MLCP with a mixture of discrete and continuous variables and introduces two relaxations: σ-complementarity and ε-integrality. Depending on the particular application one or both of these relaxations may be useful. Theorem 1 provides justification for one of the disjunctive constants (M_2) used in this MILP. Theorem 2 shows under reasonable conditions, when there exists a solution to this MILP. Section 2 also discusses some practical aspects of solving the aforementioned DC-MLCP including a heuristic for how to estimate the key complementarity relaxation constant M_1 in a general context. Note that these constants are problem specific and change with the type of application. In Theorem 4 we show how to calculate M_1 for an illustrative network example.

In Section 3, the general DC-MLCP is specialized to a market-clearing problem. Such a problem is an auction mechanism to determine which production offers and consumption bids are accepted by the market operator, which has the target of maximizing social welfare. Social welfare is computed based on producer-declared offer prices and consumers' declared bid prices, and therefore it is called "declared" social welfare. The clearing algorithm that is considered is intended for electricity markets and specifically represents the transmission (transportation) network and its capacity. The algorithm is also multi-period as it considers simultaneously the clearing of the market at several time periods (e.g., the hours of the day). Market-clearing algorithms of this type are commonly used by electricity market operators across the East Coast of the United States (http://www.pjm.com, http://www.iso-ne.com). The algorithm that we propose provides clearing prices that support market outcomes in the sense the producers that are actually producing have no

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incentive to leave the market. Note that this is so even though the proposed market-clearing formulation is non-convex and represents a new approach for this previously studied uplift problem. In other words, we propose a consistent price mechanism within a non-convex market clearing formulation.

The distinguishing features of the proposed pricing technique with respect to other procedures reported in the technical literature (e.g., [16,18,4]) are two-fold. First, the initial market-clearing problem is not manipulated to achieve prices that support market outcomes. Instead, optimality conditions of the original problem with integrality conditions relaxed are formulated and incorporated into a relaxation problem that allows realizing the tradeoff of integrality vs. complementarity, and obtaining via uplifts, prices that support market outcomes. Second, instead of using a two-step procedure (first solving the original MILP and then formulating and solving a modified LP), as indicated above, the proposed technique is single-step, and does not require altering the original problem by fixing integer variables to their optimal values to formulate a continuous problem from which prices (that support market outcomes) can be derived.

In Section 4, a stylized network equilibrium problem with multiple players is presented from [14] based on the earlier works [11,12]. This application is suitable to energy and other grid-based industries involving multiple players and a system operator. For this problem, two theoretical results are presented. In Theorem 3, under a very mild condition on the demand function, it is shown that there exists a valid bound M_1 that does not cut off any solutions. Then, in Theorem 4, given linear demand functions, a specific valid disjunctive constraints value for M_1 is presented. Both these sorts of results are application-specific but presented in a rather general network context to show how they might be done for other related problems as well as give guidance for this specific one.

After these motivating examples, in Section 5, we provide numerical experiments that validate the proposed approaches followed by conclusions and extensions in Section 6.

2. Discretely constrained mixed linear complementarity problems

2.1. Problem formulation

We consider a general, discretely constrained mixed linear complementarity problem. The formulation is as follows: given the vector $q=(q_1^T\ q_2^T)^T$ and matrix $A=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}$, find $z=(z_1^T,z_2^T)^T\in R^{n_1}\times R^{n_2}$ such that

$$0\le q_1+(A_{11}\ A_{12})\begin{pmatrix}z_1\\ z_2\end{pmatrix}\perp z_1\ge 0 \tag{1a}$$

$$0=q_2+(A_{21}\ A_{22})\begin{pmatrix}z_1\\ z_2\end{pmatrix}\quad z_2\ \text{free} \tag{1b}$$

$$(z_1)_c\in R_+,\quad c\in I_1^C,\quad (z_1)_d\in D_1\subseteq Z_+,\quad d\in I_1^D \tag{1c}$$

$$(z_2)_c\in R,\quad c\in I_2^C,\quad (z_2)_d\in D_2\subseteq Z,\quad d\in I_2^D \tag{1d}$$

where D_1, D_2 are given discrete sets of values. Also, $I_1^C\cup I_1^D$ is a partition of the indices $\{1,\ldots,n_1\}$ for z_1 and $I_2^C\cup I_2^D$ a partition of the indices $\{1,\ldots,n_2\}$ for z_2, i.e., $z_k=((z_k)_{I_k^C}^T\ (z_k)_{I_k^D}^T)^T, k=1,2$ with the continuous variables shown first, without loss of generality. As an example, suppose that the nonnegative vector z_1 has five components, i.e., $z_1=(z_{11},z_{12},z_{13},z_{14},z_{15})^T$ with the first and third constrained to be discrete and the second, fourth, and fifth continuous. In that case $I_1^D=\{1,3\}, I_1^C=\{2,4,5\}$, and if $D_1=Z_+, z_{11},z_{13}\in\{0,1,2,\ldots\}$, $z_{12},z_{14},z_{15}\in R_+$. Also note that the notation $0\le w\perp v\ge 0$ is standard shorthand in complementarity modeling to indicate that the vectors w and v are both nonnegative and their inner product is zero, i.e., $w^Tv=0$.

From here on for specificity, unless otherwise indicated, the discrete sets, $D_1=\{0,1,\ldots,N\}$ and $D_2=\{-N_1,\ldots,-1,0,1,\ldots,N_2\}$ will be assumed with N,N_1,N_2 nonnegative integers. Note that all the problem formulations in this paper assume that the problems are bounded. This is not a restrictive assumption for most real-world engineering problems where quantities are bounded by physical limits and shadow prices are often bounded by demand curves or other economic mechanisms (see Theorems 3 and 4 for a demonstration of these concepts).

First, the complementarity relationship and nonnegativity for z_1 (1a) can be recast as the following disjunctive constraints [9]:

$$0\le q_1+(A_{11}\ A_{12})\begin{pmatrix}z_1\\ z_2\end{pmatrix}\le M_1(u) \tag{2a}$$

$$0\le z_1\le M_1(1-u),\quad u_j\in\{0,1\},\quad \forall j \tag{2b}$$

where M_1 is a suitably large, positive constant and u is a vector of binary variables. The other constraints (1b) can be used as is and taking (1b) with (2) would represent a reformulation of (1) with just continuous variables z_1,z_2 allowed. If we assume that there were a solution to this version of the original problem, the existence of a solution would not necessarily be guaranteed if we imposed the discrete restrictions from (1c) and (1d). To be specific, consider the following counter-example with $A=\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix}$, $q=\begin{pmatrix}-0.2\\ 0.2\end{pmatrix}$. For z with real components, this LCP is feasible. For example, $z=(0.2,\ 2)^T$ is a solution. However, if the first component of z must be integer, i.e., $z=(z_1,z_2)^T\in\ \mathbb{Z}_+\times\mathbb{R}_+$, then this LCP is infeasible. For the LCP to be feasible, there must be an integer-valued z_1 such that $\begin{pmatrix}-0.2\\ 0.2\end{pmatrix}+\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}=\begin{pmatrix}-0.2+z_1\\ 0.2-z_1\end{pmatrix}\ge 0$ which can only be true if $0.2\le z_1\le 0.2$. Hence, there are no integer values of z_1 for which this LCP is feasible.

2.2. Relaxation of the complementarity conditions

To protect against infeasibility, relaxations on both complementarity as well as integrality are used. First, we assume that the associated LCP is at least feasible, that is, there exists (z_1,z_2) such that

$$0\le q_1+(A_{11}\ A_{12})\begin{pmatrix}z_1\\ z_2\end{pmatrix},\quad z_1\ge 0 \tag{3a}$$

$$0=q_2+(A_{21}\ A_{22})\begin{pmatrix}z_1\\ z_2\end{pmatrix},\quad z_2\ \text{free} \tag{3b}$$

Then, to relax complementarity, we introduce a nonnegative vector σ of deviations combined with the disjunctive form of the complementarity conditions to get

$$0\le q_1+(A_{11}\ A_{12})\begin{pmatrix}z_1\\ z_2\end{pmatrix}\le M_1(u)+M_1\sigma \tag{4a}$$

$$0\le z_1\le M_1(1-u)+M_1\sigma \tag{4b}$$

with $u_j\in\{0,1\},\forall j$. It is not hard to see that by adding the term $M_1\sigma$, the above complementarity is relaxed and these set of conditions are always feasible assuming that $\sigma\ge 0$ is allowed to vary. The constant M_1 can vary by constraint. Clearly, $\sigma=0$ means exact complementarity is enforced. In principle, one could also add a relaxation to always ensure that the relaxed LCP is feasible by including a term $-M_1\sigma$ instead of zero as the lower bound on the left-hand side of (4a) and (4b). In what follows, we do not