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Numerical analysis of Backward-Euler discretization for simplified magnetohydrodynamic flows

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ABSTRACT

Magnetohydrodynamics (MHD) studies consider the dynamics of electrically conducting fluids. MHD are described by a set of equations, which are a combination of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. In most terrestrial applications, MHD flows occur at low magnetic Reynolds numbers. In this study, we apply the finite element method to time-dependent MHD flows with Backward-Euler discretization at low magnetic Reynolds number. We present a comprehensive error analysis for fully discrete approximation. Finally, the effectiveness of the method is illustrated by several numerical examples.

1. Introduction

The significance of magnetohydrodynamics (MHD) has increased in recent years with respect to scientific and engineering problems, such as liquid–metal cooling of nuclear reactors [1,2], electromagnetic casting of metals [3], controlling thermonuclear fusion and plasma confinement [4,5], climate change forecasting, and sea water propulsion [6,7]. Theoretical analyzes and mathematical modeling of MHD equations can be found in [8,9]. For steady-state MHD problems, the existence, uniqueness, and finite element (FE) approximation have been described in previous studies, such as [10–12]. Variational methods and numerical approximations for solving stationary MHD equations with different boundary conditions were also studied in [13–15]. For time-dependent MHD, Schmidt [16] proposed a formulation of evolutionary MHD and established the existence of global-in-time weak solutions using the Galerkin method. Time discretization schemes for MHD problems were studied by Yuksel and Ingram [17], where they provided proofs of stability and error analyzes for semi-discrete approximations (FE in space) and fully-discrete approximation (FE in space, Crank–Nicolson time-stepping). Trenchea [18] proved the unconditional stability of a partitioned method for the evolutionary full MHD equations with a high magnetic Reynolds number. Layton et al. [19] introduced two partitioned methods to solve evolutionary MHD equations and provided a complete error analysis. An analysis of the semidiscrete approximation of the problem was presented in [17]. In the present study, we provide a stability and convergence analysis of an FE discretization for a time-dependent MHD flow at a low magnetic Reynolds number and under a quasi-static approximation with Backward-Euler time discretization.

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be an open, regular domain. The dimensionless quasi-static MHD is modeled by the following system (e.g., [17]). Given time $T > 0$, body force f, interaction parameter $N > 0$, Hartmann number $M > 0$, and by letting

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$\Omega_T := [0,T]\times\Omega$, find the velocity $u:\Omega_T\to\mathbb{R}^d$, pressure $p:\Omega_T\to\mathbb{R}$, electric current density $j:\Omega_T\to\mathbb{R}^d$, magnetic field $B:\Omega_T\to\mathbb{R}^d$, and electric potential $\phi:\Omega_T\to\mathbb{R}$ that satisfy

$$\begin{aligned} &N^{-1}(u_t+u\cdot\nabla u)=f+M^{-2}\Delta u-\nabla p+j\times B, \quad \nabla\cdot u=0,\\ &-\nabla\phi+u\times B=j, \quad \nabla\cdot j=0,\\ &\nabla\times B=R_m j, \quad \nabla\cdot B=0, \end{aligned} \tag{1}$$

subject to boundary and initial conditions

$$\begin{aligned} &u(x,t)=0, \quad \forall(x,t)\in\partial\Omega\times[0,T],\\ &\phi(x,t)=0, \quad \forall(x,t)\in\partial\Omega\times[0,T],\\ &u(x,0)=u_0(x), \quad \forall x\in\Omega. \end{aligned} \tag{2}$$

Here, $R_m = UL/\eta > 0$, U is the characteristic speed, L is the length of the problem, $\eta>0$ is the magnetic diffusivity, $u_0\in H_0^1(\Omega)^d$, and $\nabla\cdot u_0=0$. j and $\nabla\times B$ in (1) and (3a) decouple when $R_m\ll 1$. If we suppose that B is an applied (and known) magnetic field, (1) reduces to the following simplified MHD (SMHD) system (e.g., [17]). Find u,p,ϕ that satisfy

$$\begin{aligned} &N^{-1}(u_t+u\cdot\nabla u)=f+M^{-2}\Delta u-\nabla p+B\times\nabla\phi+(u\times B)\times B,\\ &\nabla\cdot u=0,\\ &-\Delta\phi+\nabla\cdot(u\times B)=0, \end{aligned} \tag{3}$$

subject to (2).

The remainder of this paper is organized as follows. In Section 2, we present notations and a weak formulation of (3), which are required for the stability and convergence analysis. In Section 3, we prove the stability analysis for fully-discrete approximation with the Backward-Euler (BE) time stepping method. We prove that the method is unconditionally stable. Moreover, for $h,\Delta t\to 0$, the order of convergence for the method is $O(\Delta t+h^r)$, where r is the order of the FE approximation. Therefore, the comprehensive error analysis and convergence of the fully-discrete approximation are proposed in Section 4. In Section 5, several examples are presented to illustrate the convergence and the effectiveness of the method.

2. Problem formulation

We denote the L^2-norm and inner product by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. The $W_p^k(\Omega)$-norm and the $W_p^k(\Omega)$-semi-norm are denoted by $\|\cdot\|_{p,k} := \|\cdot\|_{W_p^k(\Omega)}$ and $|\cdot|_{W_p^k(\Omega)}$, respectively. For $p=2$, we write $H^k(\Omega) := W_2^k(\Omega)$, and we denote $\|\cdot\|_k$ and $|\cdot|_k$ as the corresponding norm and semi-norm. We denote the pressure, velocity, and electric potential spaces by

$$\begin{aligned} &Q := \left\{q\in L^2(\Omega) : \int_\Omega q=0\right\},\\ &X := \left\{v\in H^1(\Omega)^d : v|_{\partial\Omega}=0\right\},\\ &S := \left\{\psi\in H^1(\Omega) : \psi|_{\partial\Omega}=0\right\} \end{aligned}$$

respectively. $X^* = H^{-1}(\Omega)$ is the closure of $L^2(\Omega)$ in $\|.\|_{-1}$, where

$$\|f\|_{-1} := \sup_{v\in X}\frac{(f,v)}{\|\nabla v\|}.$$

Let $L^q(0,T;W_p^k(\Omega))$ denote the space

$$L^q(0,T;W_p^k(\Omega)) = \left\{\begin{array}{ll} v:(0,T)\to W_p^k(\Omega) : v \text{ is measurable} \\ \text{and} \quad \int_0^T \|v(t)\|^q_{W_p^k(\Omega)}dt<\infty \end{array}\right\},$$

endowed with the norm

$$\|v\|_{L^q(0,T;W_p^k(\Omega))} := \left(\int_0^T \|v(t)\|^q_{W_p^k(\Omega)}dt\right)^{1/q}.$$

We write $L^q(W_p^k)=L^q(0,T;W_p^k(\Omega))$ and $C^m(W_p^k)=C^m([0,T];W_p^k(\Omega))$. For $v(x,t)$ and $1\leqslant p\leqslant\infty$, we introduce

$$\begin{aligned} \|v\|_{\infty,k} &:= \operatorname*{ess\,sup}_{0<t<T}\|v(t,\cdot)\|_k,\\ \|v\|_{p,k} &:= \left(\int_0^T\|v(t,\cdot)\|_k^p dt\right)^{1/p}. \end{aligned}$$

Let V be the divergence-free subspace of X, i.e.,

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