ARTICLE IN PRESS

Applied Mathematical Modelling xxx (2014) xxx–xxx

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Anisotropic meshing with time-stepping control for unsteady convection-dominated problems

G. Jannoun, E. Hachem *, J. Veysset, T. Coupez

MINES ParisTech, Center for Materials Forming (CEMEF), UMR CNRS 7635, BP 207, 06904 Sophia-Antipolis, France

ARTICLE INFO

Article history:
Received 24 December 2012
Received in revised form 30 July 2014
Accepted 1 October 2014
Available online xxxx

Keywords:
Anisotropic meshing
Adaptive time stepping
Convection-dominated problem
Boundary layers
Stabilized finite element method

ABSTRACT

In this work, we develop a new anisotropic space and time adaptive method for convection dominated problems with inner and boundary layers. A new route to construct a metric field directly at the nodes of the mesh is highlighted using the length distribution tensor and an edge based error analysis. A Streamline Upwind Petrov–Galerkin (SUPG) finite element method is employed to solve the unsteady convection–diffusion equation. The numerical experiments show that the use of both space and time adaptivity generates optimal time stepping, allows the recovery of the global convergence order of the numerical schemes, reduces the computational time and cost and produces accurate and oscillation free numerical solutions.

1. Introduction

The numerical solution of the unsteady convection–diffusion equation using the Galerkin formulation normally exhibits global spurious oscillations in convection-dominated problems, especially in the vicinity of sharp gradients. Over the last two decades, a variety of finite element approaches have been proposed to deal with such situations. These methods increase stability while maintaining consistency by adding weighted residual terms to the weak formulation of the problem. They have grown in popularity, especially in applications to fluid dynamics, heat transfer and fluid–structure interactions. We can find the SUPG method (Streamline Upwind Petrov–Galerkin) in [1–3], the Galerkin/least-squares (GLS) method in [4,5], the gradient Galerkin/least-squares (GGLS) method in [6,7], the unusual stabilized method (USFEM) in [8–10], the enriched method with time interpolation in [11], the subgrid scale method in [12] and many others, each was used to optimize the performance of the finite element formulations for advection–diffusion equation with or without production.

At the same time, from a practical point of view, in order to perform long-time and large-scale industrial applications it is preferable to choose a number of nodes N based on the available computational resources and determine the most accurate possible solution. So the level of accuracy is not set apriori but it is highly desirable to have the best representation of the simulated phenomena.

This goal opens the door to the emergence of many numerical techniques that aim at optimizing both the spatial and the temporal discretizations [13–23]. Indeed, combining stabilized finite element methods with these techniques allows the recovery of the global convergence order of the numerical schemes, reduces the computational time and cost and produces accurate and oscillation free numerical solutions.

* Corresponding author. Tel.: +33 (0)4 92 95 74 58; fax: +33 (0)4 92 38 97 52.
E-mail address: elie.hachem@mines-paristech.fr (E. Hachem).

http://dx.doi.org/10.1016/j.apm.2014.10.005

One of the most popular techniques in this domain is the anisotropic mesh adaptation. It allows the capture of scale heterogeneity in particular when simulating multi-components systems as the discontinuities and high gradients of the solution are captured with good accuracy for a reasonable number of elements.

Anisotropic mesh adaptation was first proposed in the late 1980s [24–27]. Significant research effort has been devoted in the last few years to devise fruitful anisotropic mesh adaptation techniques with real applications. We distinguish four major error estimates for anisotropic adaptation: the hessian based relying on the solution hessian information to evaluate the linear interpolation error [28–31], the a posteriori estimates approximating the discretization error using a theoretical analysis [32–36], the a priori error estimates [37,38] and the goal oriented estimates that provide mathematical framework for assessing the quality of some functionals [39–42]. Thanks to these technical and theoretical advances a considerable improvement was obtained in the accuracy and the efficiency of numerical simulations.

The challenging construction of the anisotropic mesh adaptation can be conveyed to that of constructing an appropriate mesh tensor by means of a discretization error analysis. For instance, during a re-meshing process, the elements of the original mesh are much more volatile than its nodes. Thus for practical reasons the meshing tools demand a nodal metric map. Other challenges are related to the possibility of constructing error estimators and adapting the mesh while taking into account different fields in particular for coupled systems (i.e. velocity, temperature, concentration,...).

In this work, we propose an alternative to these challenges. The main target of the approach is to produce extremely stretched and highly directional elements under the constraint of a fixed number of nodes. We intend to develop a metric based mesh adaptation that is capable of well capturing the anisotropy of a physical phenomena. The metric tensor would prescribe optimal mesh sizes and element orientations. It is a procedure that can be divided into three key steps. We start by constructing a length distribution tensor followed by a second order optimal nodal gradient reconstruction procedure as proposed in [43]. Then we determine the edge based error estimates that are driven entirely from an a posteriori analysis without any prior assumptions. The developed algorithm strives to improve the quality of the aforementioned error estimates by attempting to reduce and equidistribute the error over the edges of the mesh under the constraint of a fixed degree of freedom. The novelty of this paper resides in the combination of an edge-based error estimation with the equidistribution principle to derive a set of edge stretching factors resulting in an optimal anisotropic mesh adaptation. Unlike the hessian-based techniques for metric construction, the method that we propose avoids the reconstruction of this tensor and renders a reduction in the computational cost.

The second objective of this paper is to use the information gathered from the developed anisotropic mesh adaptation procedure together with the equidistribution principle in order to compute time step size that would significantly improve the computational cost and the global accuracy of the calculations. In general, a small time-step size produces more accurate solutions but with a high computational cost since more time-steps and hence more computations are required. On one hand, adopting a fixed time-stepping all over the simulation, requires an apriori knowledge of the user about the choice of this constant as it should satisfy a stability condition to guarantee the convergence of the numerical solution to the analytical one. On the other hand, determining the time-step sizes through a *CFL* argument might produce very small time-steps as their values depend on the smallest mesh-size. This choice of time-steps can be computation-wise very limiting especially when used with anisotropic meshing, as in the latter technique the mesh sizes can become very small at the locations of high solution gradients.

Using the error equidistribution in space and time, the developed mesh adaptation method, allows the control of the L^p norm of space–time interpolation error. It can be applied to any type of problems, allows a good and automatic capture of boundary and emerging inner layers in particular for high Peclet number. With such an advantage, the proposed approach recovers the orders of convergence and could be seen as a very useful tool for very stiff and convection-dominated problems. It is important to mention that the method tends to refine the mesh in the hierarchical importance of the solution's gradient. In other words, if new features (with high gradients) appear in the solution, the mesh will be automatically coarsened in regions with lower gradient and will be refined near the newly emerging features. In this case, if a small number of nodes was fixed by the user, the solution will still be well captured although not with the same degree of accuracy.

The combination of the space and time adaptive algorithms constitutes a tool to study with the available computational resources (a limitation on the number of degrees of freedom) and simulate real large unsteady convection dominated problems. The method is first demonstrated to produce optimal anisotropic meshes minimizing the L^2 norm of the interpolation error. Note that the only parameter to be chosen by the user is a fixed number of nodes that will provide a control of both space and time.

The method demonstrated second order convergence in space and time. The results were also compared with ones presented in the literature and appear in general agreement with them. Finally 2D and 3D coupled problems were considered to prove the performance of the method, its scalability and its applicability to any type of equations.

This paper is organized as follows: we start in Section 2 with the description of the developed anisotropic mesh adaptation method. Then we present in Section 3 the time adaptation technique. The Streamline Upwind Petrov Galerkin method that is used to solve the unsteady convection–diffusion equation is presented in Section 4. Section 5 contains several numerical experiments showing the efficiency and the accuracy of the proposed method. Finally, Section 6 is dedicated to the conclusion and the upcoming work.

Please cite this article in press as: G. Jannoun et al., Anisotropic meshing with time-stepping control for unsteady convection-dominated problems, Appl. Math. Modell. (2014), http://dx.doi.org/10.1016/j.apm.2014.10.005