ARTICLE IN PRESS

Applied Mathematical Modelling xxx (2014) xxx–xxx

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

An adaptive mesh strategy for singularly perturbed convection diffusion problems

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ARTICLE INFO

Article history:
Received 25 October 2013
Received in revised form 9 August 2014
Accepted 1 October 2014
Available online xxxx

Keywords:
Central finite difference schemes
Convection dominated singularly perturbed problems
Layer-adaptive meshes

ABSTRACT

In this paper, a new adaptive mesh strategy has been developed for solving convection dominated, convection–diffusion singularly perturbed problems (SPP) using second order central difference schemes. Our strategy uses a novel, entropy-like variable as the adaptation parameter for convection diffusion SPP. Further, unlike the popular layer adapted meshes mainly by Bakhvalov (B-type) and Shishkin (S-type), no pre-knowledge of the location and width of the layers (boundary as well as interior) is needed. The method is completely free of arbitrary perturbation parameters ϵ (robust) and results in oscillation free solutions to a range of convection–diffusion SPP. Numerical results for various test problems: linear (boundary layers with and without turning points), nonlinear and systems of coupled equations are presented. This method is expected to aid in robust computations of convection dominated SPP.

1. Introduction

Singularly perturbed differential equations, which have non smooth solutions with singularities related boundary layers, and their numerical solutions have been quite an active field of research over the last few decades. Starting from late 60's by Bakhvalov [1], a large number of numerical techniques have been given by many researchers. Well developed techniques are now available for the numerical solutions of outside the layers regions but solving the problem inside the layers region is limited and still under investigation. To deal with such problems, robust layers resolving methods: methods which generates numerical approximation at each point in the domain for arbitrarily small values of the singular perturbation parameter ϵ are often needed in the real world applications. Layer adapted meshes have been first proposed by Bakhvalov for reaction diffusion equations and later on by Gartland [2] and others for convection diffusion equations. In early 90s, special piecewise uniform meshes have been introduced by Shishkin [3]. Because of simple structure of Shishkin meshes they have attracted much attention and are widely used for numerically approximation of SSP. The limitation with Shishkin meshes is the requirement of a priori knowledge of the location of the layer regions. A detailed survey about layer adapted meshes for convection diffusion equations can be found in [4].

It is well known that whenever central finite difference schemes (second order or higher order central schemes, non-monotone methods) are applied to solve convection diffusion singularly perturbed problems numerically on uniform meshes, unphysical oscillations are observed in the numerical solution and their magnitude increases in the layers

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http://dx.doi.org/10.1016/j.apm.2014.10.019
0307-904X/

Please cite this article in press as: V. Kumar, B. Srinivasan, An adaptive mesh strategy for singularly perturbed convection diffusion problems, Appl. Math. Modell. (2014), http://dx.doi.org/10.1016/j.apm.2014.10.019

(boundary and interior) regions. The presence of these large unphysical oscillations in the approximated solution shows that central finite difference operators on the uniform meshes are unstable below a certain value of the singular perturbation parameter. To eliminate these oscillations while retaining the order of accuracy, one needs either a fine mesh at the layer. This maybe done either via uniformly fine meshing or via an adaptive mesh strategy. The former strategy increases significantly in computational cost as the perturbation parameter ϵ decreases. In this paper, we exploit such oscillations indirectly to locate the layers regions and then generate the layer adaptive meshes, via an entropy production operator which is positive automatically whenever the solution is unphysical, which effectively corresponds to insufficient resolution. We restrict ourselves to the class of SPPs which correspond to second order boundary value problems with a non-zero coefficient of the convection (first derivative) term.

Section 2 has been dedicated to adaptive mesh strategies and various numerical results have been discussed in Section 3 followed by the conclusion.

2. Adaptive mesh strategy

2.1. Linear convection diffusion problems

We consider the following one dimensional steady-state convection diffusion problem as

$$\mathcal{L}u(x) = -\epsilon u''(x) - a(x)u'(x) + b(x)u(x) = f \in (0,1), \quad u(0) = \alpha, \quad u(1) = \beta \tag{2.1}$$

with a small positive perturbation parameter ϵ. The above equation physically represents the steady state of transport of a passive scalar by a background velocity. For instance, $u(x)$ could represent the temperature and $a(x)$ could represent the (possibly spatially varying) background velocity and ϵ is the diffusion coefficient. As ϵ becomes smaller, convective processes start dominating over the diffusive processes [5]. Without loss of generality, we consider $b(x) \geqslant 0$ and $a(x)$ can change the sign (as in the case of turning point problem) in the given domain $[0,1]$. For $a(x) \geqslant \gamma > 0$, the above problem has unique solution and exhibit a boundary layer of $O(\epsilon)$ at $x = 0$. It was proved by Kellogg and Tsan [6] that u and its derivatives up to an arbitrary prescribed order q (depending on the smoothness of the data) can be bounded by

$$|u^{(k)}(x)| \leqslant C\{1 + \epsilon^{-k}e^{-\gamma x/\epsilon}\}, \quad \text{for } k = 0,1,\ldots,q \quad \text{and} \quad x \in [0,1],$$

where C denotes a generic positive constant independent of ϵ and the number of mesh points used.

We now define an auxiliary equation – the entropy production equation – by multiplying with an appropriate test function. From the case of scalar hyperbolic conservation laws [7], it is well known that for scalar conservation laws, u^2 is always an appropriate entropy variable and, therefore, $2u(x)$ is a suitable multiplying test function. On multiplying (component wise), we obtain

$$\mathcal{L}u * 2u = f * 2u; \tag{2.2}$$

after performing the calculation we can write the above Eq. (2.2) as

$$-\epsilon(S'' - 2(u')^2) - aS' + 2bS - 2uf = 0 \text{ where } S = u^2. \tag{2.3}$$

Rewriting the above equation, we get

$$-\epsilon S'' - aS' - 2uf = -2bS - 2\epsilon(u')^2. \tag{2.4}$$

We label the linear operator on the left hand side (LHS) in the above Eq. (2.4), as the entropy production operator with analogy to similar operators in hyperbolic conservation laws [7]. The continuous operator should obviously be negative for all $x \in [0,1]$ (as right hand side (RHS) is always negative for all $x \in [0,1]$).

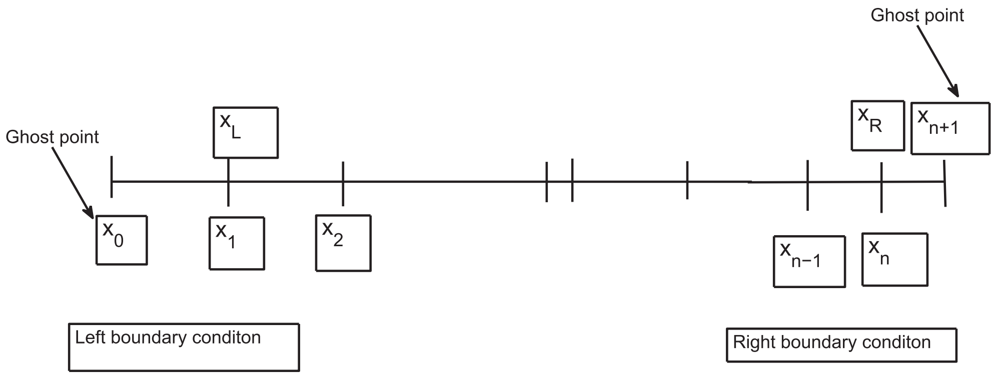

Fig. 1. Ghost points at the left and the right boundaries.