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Short Communication

On the propagation of long thickness-stretch waves in piezoelectric plates

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ABSTRACT

We study the propagation of thickness-stretch waves in a piezoelectric plate of polarized ceramics with thickness poling or crystals of class 6 mm whose sixfold axis is along the plate thickness. For device applications we consider long waves with wavelengths much longer than the plate thickness. A system of two-dimensional equations in the literature governing thickness-stretch, extensional, and symmetric thickness-shear motions of the plate is further simplified. The equations obtained can be used to analyze piezoelectric plate acoustic wave devices operating with thickness-stretch modes.

1. Introduction

Polarized ferroelectric ceramics are transversely isotropic and exhibit piezoelectric coupling. Ceramic plates with thickness poling (see Fig. 1 where the x_2 axis is determined from x_3 and x_1 by the right-hand rule) are common structures for piezoelectric transducers [1]. When such a plate is electroded at the top and bottom with an alternating driving voltage applied across the electrodes, through the piezoelectric constant e_{33}, the plate can be excited into thickness-stretch (TSt) [2] vibrations with a dominant displacement component u_3. Piezoelectric crystals of class 6 mm like ZnO and AlN are of current and growing interest for thin film bulk acoustic wave resonators (FBARs) [3]. The mathematical structures of the material tensors/matrices of crystals of class 6 mm are the same as those of polarized ceramics. Therefore piezoelectric plates made from crystals of class 6 mm with the sixfold axis along the plate thickness can also be excited into TSt motions by a thickness electric field.

Strictly speaking, pure TSt motions are only possible in unbounded plates. Then the three-dimensional (3-D) theory of piezoelectricity reduces to a one-dimensional (1-D) model with one spatial coordinate x_3 only along the plate thickness [4]. In real plate piezoelectric devices of finite sizes, the TSt operating modes have slow, in-plane variations along x_1 and x_2 and are called transversely varying TSt modes. They are long plate waves whose in-plane wavelengths are much longer than the plate thickness. Among TSt waves of various orders, the fundamental one that has one nodal plane with zero displacement at the plate middle plane is the one that is most useful in applications. Due to the in-plane field variation, Poisson's effect and edge effects, the fundamental TSt wave is coupled to the in-plane extensional wave and the symmetric thickness-shear (TSh) wave [5]. Analyzing these three coupled waves using the 3-D theory of piezoelectricity presents considerable mathematical challenges. Therefore, a system of two-dimensional (2-D) equations for coupled TSt, symmetric TSh and extensional motions in piezoelectric plates were derived in [5]. These equations depend on the two in-plane spatial coordinates x_1 and x_2 only. This reduces the spatial dimension by one and simplifies the problem significantly.

The equations in [5], although much simpler than the 3-D equations, are still fairly complicated. An approximate procedure for similar waves in elastic plates suggests that a further simplification and specialization of the equations in [5] is possible. Coupled TSt, symmetric TSh and extensional motions in isotropic elastic plates can be described by 2-D theories due to Mindlin and Medick [6] or Lee and Nikodem [7]. Then the symmetric TSh and extensional displacements which are of minor interest can be eliminated through a procedure called TSt approximation [8], resulting in a single 2-D equation governing the TSt displacement of interest. This is parallel to the well-known TSh approximation in [9] for the elimination of the flexural displacement from the coupled theory for the fundamental TSh and flexure to obtain simple equations for the fundamental TSh modes. In view of the wide use of piezoelectric

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Fig. 1. A piezoelectric plate and coordinate system.

transducers and FBARs operating with the fundamental TSt waves of piezoelectric plates, in this paper we extend the TSt approximation in [8] from elastic plates to piezoelectric plates by simplifying the system of 2-D equations in [5].

2. Two-dimensional equations for coupled TSt, symmetric TSh, and extension

In this section we summarize the relevant equations from [5]. Consider the ceramic plate shown in Fig. 1 whose thickness is $2b$ and mass density is ρ. The top and bottom surfaces of the plate are electroded, with the electrodes represented by the thick lines in the figure. The thickness and mass density of the electrodes are $2b'$ and ρ'. To derive 2-D plate equations, the variations of the mechanical displacement u and the electric potential ϕ along the plate thickness x_3 are approximated by the following simple and known functions in [5]:

$$\begin{aligned} u_1(x_1,x_2,x_3,t) &\cong u_1^{(0)}(x_1,x_2,t)+u_1^{(2)}(x_1,x_2,t)\cos[\pi(1-x_3/b)],\\ u_2(x_1,x_2,x_3,t) &\cong u_2^{(0)}(x_1,x_2,t)+u_2^{(2)}(x_1,x_2,t)\cos[\pi(1-x_3/b)],\\ u_3(x_1,x_2,x_3,t) &\cong u_3^{(1)}(x_1,x_2,t)\cos\left[\tfrac{\pi}{2}(1-x_3/b)\right], \end{aligned} \quad (1)$$

$$\phi(x_1,x_2,x_3,t) \cong \overline{V}_0(t)+\overline{V}_1(t)\frac{x_3}{b}+\phi^{(2)}(x_1,x_2,t)\sin[\pi(1-x_3/b)], \quad (2)$$

where $u_1^{(0)}$ and $u_2^{(0)}$ are the plate extensional displacements, $u_1^{(2)}$ and $u_2^{(2)}$ are the plate symmetric TSh displacements, and $u_3^{(1)}$ is the plate TSt displacement. $\overline{V}_0$ and $\overline{V}_1$ are determined by the applied voltage across the electrodes. $\phi^{(2)}$ is a second-order electric potential whose variation along the plate thickness is more complicated than linear. From [5], the equations governing the above five plate displacement components and $\phi^{(2)}$ are

$$\begin{aligned} &(c_{11}^{(0)}+c_{12}^{(0)})\nabla\left(\nabla\cdot\mathbf{u}_T^{(0)}\right)+2c_{66}\nabla^2u_T^{(0)}+\tfrac{2}{b}c_{13}\nabla u_3^{(1)}-c_{11}^{(3)}\nabla\left(\nabla\cdot\mathbf{u}_T^{(2)}\right)\\ &\quad+\tfrac{\pi}{b}e_{31}^{(3)}\nabla\phi^{(2)}=2(1+R)\rho\ddot{\mathbf{u}}_T^{(0)}+2R\rho\ddot{\mathbf{u}}_T^{(2)}, \end{aligned} \quad (3)$$

$$\begin{aligned} &c_{55}^{(1)}\nabla^2u_3^{(1)}-\tfrac{2}{b}c_{13}\nabla\cdot\mathbf{u}_T^{(0)}+\tfrac{2}{3b}(c_{13}+4c_{55})\nabla\cdot\mathbf{u}_T^{(2)}-\tfrac{\pi^2}{4b^2}c_{33}u_3^{(1)}\\ &\quad+\tfrac{8}{3\pi}e_{15}\nabla^2\phi^{(2)}-\tfrac{2\pi}{3b^2}e_{33}\phi^{(2)}-\tfrac{2}{b^2}e_{33}\overline{V}_1=(1+2R)\rho\ddot{\mathbf{u}}_3^{(1)}, \end{aligned} \quad (4)$$

$$\begin{aligned} &\frac{c_{11}^{(2)}+c_{12}^{(2)}}{2}\nabla\left(\nabla\cdot\mathbf{u}_T^{(2)}\right)+c_{66}\nabla^2\mathbf{u}_T^{(2)}-c_{11}^{(3)}\nabla^2\mathbf{u}_T^{(0)}-\frac{2}{3b}(c_{13}+4c_{55})\nabla u_3^{(1)}\\ &\quad-\tfrac{\pi^2}{b^2}c_{55}\mathbf{u}_T^{(2)}-\tfrac{\pi}{b}(e_{31}^{(2)}+e_{15})\nabla\phi^{(2)}=(1+2R)\rho\ddot{\mathbf{u}}_T^{(2)}+2R\rho\ddot{\mathbf{u}}_T^{(0)}, \end{aligned} \quad (5)$$

$$\begin{aligned} &\tfrac{\pi}{b}\left(e_{15}+e_{31}^{(2)}\right)\nabla\cdot\mathbf{u}_T^{(2)}-\tfrac{\pi}{b}e_{31}^{(3)}\nabla\cdot\mathbf{u}_T^{(0)}+\tfrac{8}{3\pi}e_{15}\nabla^2u_3^{(1)}-\tfrac{2\pi}{3b^2}e_{33}u_3^{(1)}\\ &\quad-\varepsilon_{11}\nabla^2\phi^{(2)}+\tfrac{\pi^2}{b^2}\varepsilon_{33}^{(2)}\phi^{(2)}-\tfrac{\pi}{b^2}\varepsilon_{33}^{(3)}\overline{V}_1=0, \end{aligned} \quad (6)$$

where the in-plane extensional and symmetric TSh displacement vectors, the in-plane 2-D Laplacian, and the electrode-plate mass ratio are denoted by

$$\begin{aligned} &\mathbf{u}_T^{(n)}=u_1^{(n)}\mathbf{e}_1+u_2^{(n)}\mathbf{e}_2,\quad n=0,2,\\ &\nabla=\mathbf{e}_1\tfrac{\partial}{\partial x_1}+\mathbf{e}_2\tfrac{\partial}{\partial x_2},\quad R=\tfrac{2\rho'b'}{\rho b}. \end{aligned} \quad (7)$$

In (3)–(6), $c_{pq}^{(0)}$, $c_{pq}^{(1)}$, $c_{pq}^{(2)}$, $c_{pq}^{(3)}$, $e_{kp}^{(2)}$, $e_{kp}^{(3)}$, $\varepsilon_{kl}^{(2)}$ and $\varepsilon_{kl}^{(3)}$ are plate material constants of various orders. They are complicated functions of the usual elastic, piezoelectric, and dielectric constants [5].

3. Elimination of symmetric TSh and extension

Our goal is to obtain a simple 2-D equation for the main TSt displacement $u_3^{(1)}$. Since TSt and the in-plane extension are inherently coupled through Poisson's effect, we cannot simply set the extensional and symmetric TSh displacements to zero. Instead, we eliminate them through a procedure based on two approximations. One is that we consider long TSt waves with small wave numbers. The other is that we are in the frequency range very close to the fundamental TSt frequency ω_0 of the plate. For long waves we neglect the second-order in-plane spatial derivatives in (3) and (5) [8] because a differentiation with respect to x_1 or x_2 translates into a multiplication with a small in-plane wave number for long waves. Therefore, for long waves (3) and (5) become

$$\frac{2}{b}c_{13}\nabla u_3^{(1)}+\frac{\pi}{b}e_{31}^{(3)}\nabla\phi^{(2)}=2(1+R)\rho\ddot{\mathbf{u}}_T^{(0)}+2R\rho\ddot{\mathbf{u}}_T^{(2)}, \quad (8)$$

$$\begin{aligned} &-\frac{2}{3b}(c_{13}+4c_{55})\nabla u_3^{(1)}-\frac{\pi^2}{b^2}c_{55}\mathbf{u}_T^{(2)}-\frac{\pi}{b}\left(e_{31}^{(2)}+e_{15}\right)\nabla\phi^{(2)}\\ &\quad=(1+2R)\rho\ddot{\mathbf{u}}_T^{(2)}+2R\rho\ddot{\mathbf{u}}_T^{(0)}. \end{aligned} \quad (9)$$

For time-harmonic motions at frequencies very close to the fundamental TSt frequency ω_0, we let $\mathbf{u}_T^{(0)}\cong\mathbf{u}_T^{(0)}\exp(i\omega_0t)$ and $\mathbf{u}_T^{(2)}\cong\mathbf{u}_T^{(2)}\exp(i\omega_0t)$ in (8) and (9) [8] to obtain

$$2(1+R)\rho\,\mathbf{u}_T^{(0)}+2R\rho\,\mathbf{u}_T^{(2)}=-\frac{2}{b\omega_0^2}c_{13}\nabla u_3^{(1)}-\frac{\pi}{b\omega_0^2}e_{31}^{(3)}\nabla\phi^{(2)}, \quad (10)$$

$$\begin{aligned} 2R\rho\,\mathbf{u}_T^{(0)}+\left[(1+2R)\rho-\frac{\pi^2}{b^2\omega_0^2}c_{55}\right]\mathbf{u}_T^{(2)}&=\frac{2}{3b\omega_0^2}(c_{13}+4c_{55})\nabla u_3^{(1)}\\ &\quad+\frac{\pi}{b\omega_0^2}(e_{31}^{(2)}+e_{15})\nabla\phi^{(2)}. \end{aligned} \quad (11)$$

Then we take the divergence of both sides of (10) and (11). This leads to two linear equations for $\nabla\cdot\mathbf{u}_T^{(0)}$ and $\nabla\cdot\mathbf{u}_T^{(2)}$. Solving these two equations we obtain

$$\begin{aligned} \nabla\cdot\mathbf{u}_T^{(0)}&=A_1^{(1)}\cdot\nabla^2u_3^{(1)}+A_2^{(1)}\cdot\nabla^2\phi^{(2)},\\ \nabla\cdot\mathbf{u}_T^{(2)}&=A_1^{(2)}\cdot\nabla^2u_3^{(1)}+A_2^{(2)}\cdot\nabla^2\phi^{(2)}, \end{aligned} \quad (12)$$

where

$$\begin{aligned} A_1^{(1)}&=-\tfrac{2}{3b\omega_0^2A}\left[3c_{13}\rho+8R\rho(c_{13}+c_{55})-3c_{13}c_{55}\tfrac{\pi^2}{b^2\omega_0^2}\right],\\ A_2^{(1)}&=-\tfrac{\pi\rho}{b\omega_0^2A}\left[e_{31}^{(3)}\left((1+2R)-\tfrac{\pi^2}{b^2\omega_0^2}c_{55}\right)+2R(e_{31}^{(2)}+e_{15})\right],\\ A_1^{(2)}&=\tfrac{4\rho}{3b\omega_0^2A}[(1+4R)c_{13}+(1+R)4c_{55}],\\ A_2^{(2)}&=\tfrac{2\rho\pi}{b\omega_0^2A}[(1+R)(e_{31}^{(2)}+e_{15})+Re_{31}^{(3)}], \end{aligned} \quad (13)$$

$$A=\begin{vmatrix} 2(1+R)\rho & 2R\rho\\ 2R\rho & (1+2R)\rho-\tfrac{\pi^2}{b^2\omega_0^2}c_{55} \end{vmatrix}. \quad (14)$$

Substituting (12) back into (4) and (6), we obtain the following two equations for $u_3^{(1)}$ and $\phi^{(2)}$:

$$\begin{aligned} &[c_{55}^{(1)}-\tfrac{2}{b}c_{13}A_1^{(1)}+\tfrac{2}{3b}(c_{13}+4c_{55})A_1^{(2)}]\nabla^2u_3^{(1)}-\tfrac{\pi^2}{4b^2}c_{33}u_3^{(1)}\\ &\quad+[-\tfrac{2}{b}c_{13}A_2^{(1)}+\tfrac{2}{3b}(c_{13}+4c_{55})A_2^{(2)}+\tfrac{8}{3\pi}e_{15}]\nabla^2\phi^{(2)}\\ &\quad-\tfrac{2\pi}{3b^2}e_{33}\phi^{(2)}-\tfrac{2}{b^2}e_{33}\overline{V}_1=\kappa\rho(1+2R)\ddot{\mathbf{u}}_3^{(1)}, \end{aligned} \quad (15)$$

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