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Computational uncertainty quantification for some strongly degenerate parabolic convection–diffusion equations

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A B S T R A C T

Strongly degenerate parabolic convection–diffusion equations arise as governing equations in a number of applications such as traffic flow with driver reaction and anticipation distance and sedimentation of solid–liquid suspensions in mineral processing and wastewater treatment. In these applications several parameters that define the convective flux function and the degenerating diffusion coefficient are subject to stochastic variability. A method to evaluate the variability of the solution of the governing partial differential equation in response to that of the parameters is presented. To this end, a generalized polynomial chaos (gPC) expansion of the solution is approximated by its projection onto a finite-dimensional space of piecewise polynomial functions defined on a suitable discretization of the stochastic domain, according to the basic principle of the hybrid stochastic Galerkin (HSG) approach. This approach is combined with a finite volume (FV) method, resulting in a so-called FV–HSG method, to compute the sought deterministic coefficient functions of the truncated polynomial-chaos-based expansion of the solution. Since the stochastic parameter space is now spanned by piecewise polynomial functions, one may employ the numerical result to compute the reconstruction of the numerical solution for arbitrary values of the random variables. The expectation, the variance or other stochastic quantities of the solution (as functions of time and position) can also be computed from these coefficient functions. The method is illustrated by a number of numerical examples.

1. Introduction

1.1. Scope

This work is focused on numerical methods for the quantification of the stochastic variability of solutions $u = u(x, t)$ of the strongly degenerate parabolic equation

$$\partial_t u + \partial_x f(u) = \partial_x^2 A(u), \quad (x, t) \in Q_T := I \times (0, T), \quad T > 0, \tag{1}$$

that arises from uncertainty in the parameters that define the function $a = a(u)$, where

$$A(u) = \int_0^u a(s)\,\mathrm{d}s, \quad a \in L^1[0, u_{\max}], \quad a(u) \geq 0 \quad \text{for } 0 \leq u \leq u_{\max}. \tag{2}$$

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Here the x-interval is either $I = \mathbb{R}$ corresponding to an initial value problem, or $I = (0, 1)$, for which (1) is posed with suitable initial and boundary conditions. We assume that f is a piecewise smooth, Lipschitz continuous, non-negative function with support on $(0, u_{\max})$, where $u_{\max}$ is a maximum solution value. We allow that $a(u) = 0$ on u-intervals of positive lengths, which motivates why (1) is called *strongly degenerate*. For the u-values such that $a(u) = 0$, (1) degenerates into the first-order conservation law

$$\partial_t u + \partial_x f(u) = 0, \tag{3}$$

where the location of the type-change interface is unknown beforehand. Consequently, solutions of (1) are in general discontinuous, and must be defined as weak solutions along with an entropy condition, that is, as entropy solutions.

Under the assumption of strong degeneracy, (1) arises in a number of applications, including a model of vehicular traffic with reaction times and anticipation lengths [1–3] and a model of sedimentation of flocculated suspensions [4,5]. In both applications, it is frequently assumed that

$$a(u) \begin{cases} = 0 & \text{for } u \le u_{\mathrm{c}} \text{ and } u > u_{\max}, \\ > 0 & \text{for } u_{\mathrm{c}} < u < u_{\max}, \\ \ge 0 & \text{for } u = u_{\max}, \end{cases} \tag{4}$$

where $u_{\mathrm{c}} \ge 0$ is a given critical value, so that (1) degenerates into (3) wherever $u \le u_{\mathrm{c}}$. The value of u_{c} is, however, problem-dependent and usually not based on first principles. It either estimates the "threshold" value of the density u beyond which nonlinear diffusive effects become significant (a motivation that is common in traffic flow modelling) or represents a "phenomenological" parameter that characterizes geometrically complicated behaviour (such as the formation of a porous network formed by sedimentation of flocculated, non-spherical particles). In any case, the value of u_{c} is subject to uncertainty, and there is theoretical and practical interest in quantifying the uncertainty in the solution of (1) in terms of that of u_{c} and other parameters that arise in the algebraic definition of $a(u)$. It is the purpose of this contribution to provide a computational method to solve this task. To this end, we introduce an appropriate definition of several random parameters, which represent uncertainty in the model problems. Based on this definition we provide the hybrid stochastic Galerkin (HSG) discretization for uncertain strongly parabolic degenerate problems. The HSG method is an intrusive stochastic Galerkin discretization method that was successfully applied to several non-linear hyperbolic problems, for example in [6,7]. In general, intrusive SG discretizations transform the underlying partial differential equation (PDE), which is assumed to depend on random parameters, into a deterministic system by means of a Galerkin projection onto the stochastic space. We present an appropriate numerical scheme, which is based on central upwind method, and apply it to several examples motivated by real-world applications. Moreover, we study the accuracy of the method in short- and long-time numerical simulations and also the influence of the several random parameters on expectation and variance of the solution.

1.2. Related work

To put the present paper into the proper perspective, we mention that the applications motivating strongly degenerate parabolic equation (1), (2), namely traffic flow and sedimentation of flocculated suspensions, are broadly discussed in [1–3,8,9] and [5,10], respectively (see also references cited in these papers). For the application to sedimentation, (1) equipped with initial and zero-flux boundary conditions describes the simple process of batch settling in a column, but the same functions f and A also arise in more involved models of continuous sedimentation in clarifier–thickener models, in which the governing PDE (not written out here) includes additional transport terms accounting for bulk flows, singular source terms, and discontinuous coefficients (see [4,11,12] and references cited in these papers). The usefulness of (1) as a practical model for real-world phenomena depends critically on calibration, that is, the possibility to identify the model functions f and A (or equivalently, a) for real situations. These issues are addressed in [8] and [13–16] for the models of traffic flow and sedimentation, respectively. The present work deals with a closely related issue, namely the assessment of the variability of the model prediction in response to the uncertainty in constitutive model functions. Another problem that arises for clarifier–thickener models with time-dependent control functions lies in the fact that there is not only uncertainty in the appropriate choice of material specific constitutive functions (as is considered herein), but that also many input parameters that represent time-dependent operating conditions cannot be described with deterministic accuracy but by stochastic methods. For instance, in mineral processing the uncertainty comes from the fact that the feed flow stems from other units that are not under control of the CT operator, while in wastewater treatment weather conditions, which may affect the operation of the unit, are unpredictable. An HSG approach to computationally quantify the uncertainty arising in this situation was applied in [6,7,17].

With respect to uncertainty quantification for conservation laws and related partial differential equations in a general context, we mention that the straightforward Monte Carlo (MC) computations of sampling solutions produced under stochastic variation of the input data are easily implemented, but quantifying randomness via the MC approach can be computationally inefficient due to the slow convergence rate of stochastic approach, in particular in the case of high computational costs for each sample. However, the computational efficiency of MC can be significantly improved by multilevel Monte Carlo [18,19] and quasi-Monte Carlo [20] techniques.

In this paper we focus on the hybrid stochastic Galerkin (HSG) discretization which belongs to the broad class of intrusive stochastic Galerkin (SG) methods. The application of the intrusive SG methods to the uncertainty quantification of PDEs