Structural Safety 76 (2019) 123–134

Contents lists available at ScienceDirect

Structural Safety

journal homepage: www.elsevier.com/locate/strusafe

Adaptive scaled unscented transformation for highly efficient structural reliability analysis by maximum entropy method

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ARTICLE INFO

Keywords:
Maximum entropy method
Fractional moments
Adaptive scaled unscented transformation
Correlation
High dimensionality

ABSTRACT

The approximation of the probability density function (PDF) of the performance function, especially for the tail distribution, is of paramount importance in structural reliability analysis. In this paper, a new method is proposed to derive the PDF of the performance function with accuracy and high efficiency. The derivation is based on the maximum entropy method(MEM), where the fractional moments are adopted as constraints. Since the MEM dose not involve deterministic structural analysis, the efficiency and accuracy of the proposed method is dependent on the evaluation of fractional moments. In this regard, an adaptive scaled unscented transformation (ASUT) is developed to obtain the fractional moments with only a few of sample evaluations. The proposed ASUT is applicable to problems with correlated/uncorrelated random variables. Besides, it can circumvent the so-called "curse of dimensionality" to some extent. Thus, the proposed method could be taken as a general tool for highly efficient structural reliability analysis. Numerical examples involving explicit and implicit performance functions are used to illustrate the implementation of the proposed method, which shows the great efficacy of the proposed method, particularly for a high-dimensional problem. The problems to be further investigated are also pointed out.

1. Introduction

Structural reliability analysis entails the computation of the probability that is out of the safe domain, which provides a quantitative basis of the assessment of the safety level of a structure. Although such a definition of structural reliability is quite simple, inherent difficulties still arises in reality due to the implicit nature and high nonlinearity of the performance function [1]. In this regard, approximate methods for structural reliability analysis are of great necessity.

The most commonly used methods to calculate the failure probability are the simulation methods [2–6], which involves the sampling of basic random variables and then simulating the performance function repeatedly. However, they are often too much computationally demanding to be implemented for practical applications. Alternatively, the first- or the second- order reliability method (FORM/SORM) [7–10] has been successfully applied for structural reliability computations. Due to the requirement of derivatives and sensitivities of the performance function, there could be some difficulties to employ these methods to evaluate failure probabilities when complicated and implicit performance functions are involved. In other words, the merging of FORM/SORM with finite-element analysis is not straightforward especially when the nonlinear problems are addressed [1]. The response surface method, which replaces the original performance function by a surrogate model, can significantly reduce the computational effort for structural reliability analysis. Typical surrogate models include quadratic response surfaces [11], support vector machines [12,13], neural network [14] and kriging [15], etc. Nevertheless, it is often quite difficult to quantify the error made by such a substitution [16]. The method of moments, which requires neither the computation of sensitivities nor the calculation of derivatives of the performance function, is an effective method to perform finite-element based reliability analysis [17]. However, the computation of high-order moments is always quite difficult [18], especially when the variations of basic random variables are large and nonlinearity is considered in the performance function.

Recently, the maximum entropy method (MEM) with fractional moments as constraints has received its popularity in structural reliability analysis [19,20]. The reason of this interest is that the MEM can provide an unbiased estimation of the probability density function (PDF) of the performance function and a few of fractional moments constraints are adequate in MEM. That is because the information of a large number of central moments can be embodied in a single fractional

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https://doi.org/10.1016/j.strusafe.2018.09.001
Received 17 January 2018; Accepted 2 September 2018
0167-4730/

moment. Besides, the tail distribution can be better reproduced by using fractional moments instead of integer moments as constraints in MEM [19], which is of critical significance for structural reliability computations. The efficiency and accuracy of this method relies on the numerical computation of fractional moments. For this purpose, several methods have been developed, such as the multiplicative dimension reduction method (M-DRM) [19], the unequal weighted quasi-Monte Carlo method [21] and the rotational quasi-symmetric point method (RQ-SPM) [22], etc. Unfortunately, these methods may be incapable of dealing with problems with correlated random variables and high-dimensionality, which widely exist in engineering practices and are open challenges in reliability community. Consequently, developing a highly efficient and accurate method for general structural reliability analysis, which is applicable to independent/dependent random variables, low/high-dimensional problems, is of great necessity.

The objective of this paper is to propose a highly efficient method for both fractional moments assessment and structural reliability analysis based on MEM. The paper is organized as follows. Section 2 recalls the basic of MEM with fractional moments as constraints for structural reliability analysis. In Section 3, a new adaptive scaled unscented transformation, which can accurately evaluate the fractional moments with high efficiency, is developed. Section 4 provides different numerical examples to validate the proposed method. The final section contains some concluding remarks and problems to be further investigated.

2. Maximum entropy method with fractional moments as constraints for structural reliability analysis

The performance function or the limit state function for structural reliability analysis can be expressed as

$$Z = G(\mathbf{X}) \tag{1}$$

where $\mathbf{X} = [X_1, X_2, ..., X_d]^T$ is the d-dimensional dependent/independent input random variables and G is a deterministic operator.

Then, the failure probability p_f can be expressed as

$$p_f = \Pr[Z \leqslant 0] = \int_{-\infty}^{0} p_Z(z)dz \tag{2}$$

where Pr denotes for probability for short and $p_Z(z)$ is the PDF of Z.

It is noted that once the entire range of $p_Z(z)$ is obtained, the failure probability can be evaluated by a simple one-dimensional integral. In this regard, the method, which can accurately derives the $p_Z(z)$ is of great concern. The MEM with fractional moments as constraints has been proven as an effective method for this purpose [19,23,21,22].

Since a positive random variable is concerned in MEM with fractional moments as constraints, a coordinate transformation is first implemented. According to computational experiences, the fractional operations work quite well on the order of 10^3. First, let $Z_1 = 1000 \times [Z/Z_{\max}]$, where $Z_{\max}$ is the upper bound for Z and $Z_1 \in (-1000, 1000)$. Then, let $Y = Z_1 - Z_{1,min}$, where $Z_{1,min}$ is the lower bound for Z_1 and Y is a continuous-valued positive random variable whose distribution domain is about $\Omega_Y = (0, 2000)$. The reason of performing such a transformation is that the MEM with fractional moments is applicable to positive random variable only and the same initial conditions may be employed in MEM for different problems since the distribution domain of Y could be almost the same. It is obvious that when the distribution of Y is available, the PDF of Z can be obtained easily according to the probability theory [24]. Then, the task changes to derive the PDF of Y, which is denoted as $p_Y(y)$.

The differential entropy of $p_Y(y)$, denoted as $\mathscr{H}[p_Y(y)]$, can be expressed as [25]

$$\mathscr{H}[p_Y(y)] = -\int p_Y(y)\ln[p_Y(y)]dy \tag{3}$$

and its fractional moments constraints are

$$m_Y^{\rho_k} = \int (Y)^{\rho_k} p_Y(y)dy, \quad k = 1, 2, ..., n_k \tag{4}$$

where $m_Y^{\rho_k}$ is the ρ_k-th order fractional moment of Y and ρ_ks, $k = 1, 2, ..., n_k$ denote the fractional orders, which are real numbers.

The reason of using fractional moments lies in that a single fractional moment actually contains the information about a large number of central moments so that a couple of fractional moments could be adequate to recover the PDF. This could be found from the Taylor series expansion of $(Y)^{\rho_k}$ about its mean Y_0, which could be expressed as [22]

$$\begin{aligned}(Y)^{\rho_k} = &\frac{Y_0^{\alpha}}{0!} + \frac{(Y^{\rho_k})'_{|Y=Y_0}}{1!}(Y-Y_0) + \frac{(Y^{\rho_k})''_{|Y=Y_0}}{2!}(Y-Y_0)^2 \\ &+ \cdots + \frac{(Y^{\rho_k})^{(i)}_{|Y=Y_0}}{i!}(Y-Y_0)^i + \cdots = \frac{Y_0^{\rho_k}}{0!} + \frac{\rho_k}{1!}Y_0^{\rho_k-1}(Y-Y_0) \\ &+ \frac{\rho_k \times (\rho_k-1)}{2!}Y_0^{\rho_k-2}(Y-Y_0)^2 \\ &+ \cdots + \frac{\rho_k \times (\rho_k-1)\times\cdots\times(\rho_k-i+1)}{i!}Y_0^{\rho_k-i}(Y-Y_0)^i \\ &+ \cdots = \sum_{i=0}^{\infty}\binom{\rho_k}{i}Y_0^{\rho_k-i}(Y-Y_0)^i\end{aligned} \tag{5}$$

where the binomial coefficient is

$$\binom{\rho_k}{i} = \frac{\rho_k!}{i!(\rho_k-i)!} = \frac{\rho_k \times (\rho_k-1)\times...\times(\rho_k-i+1)}{i!} \neq 0\rho_k \in \mathbb{R}\backslash\mathbb{N} \tag{6}$$

and when $i \to \infty$, $\binom{i}{\rho_k} \to 0$. For example, when $i = 10000$ and $\rho_k = 0.5$, we have

$$\begin{aligned}\binom{\rho_k}{i} = &\frac{0.5 \times (-0.5) \times (-1.5)...\times(-9999.5)}{1 \times 2 \times 3...\times 10000} = 0.5 \times (-0.25) \times (-0.5) \\ &\times ....\times(-0.99995) \approx 0\end{aligned} \tag{7}$$

In that regard, Eq. (6) can be truncated and there exists [19]

$$\mathscr{E}[(Y)^{\rho_k}] = \sum_{i=0}^{n_c}\binom{\rho_k}{i}(Y_0)^{\rho_k-i}\mathscr{E}[(Y-Y_0)^i] \tag{8}$$

where n_c is the number of truncated terms, $\mathscr{E}$ denotes the expectation operator and $\mathscr{E}[(Y-Y_0)^i]$ is the i-th central moments of the random variable Y. From Eq. (8), it is clear that a single fractional moment indeed embodies the information about a large number of central moments.

According to MEM, the PDF, which maximizes the entropy, is the most probable PDF from all the PDFs under the moments constraints. In this regard, when the fractional moments are specified as the constraints, the interested PDF $p_Y(y)$ can be derived such that

$$\begin{cases} \text{Find} \quad p_Y(y) \\ \text{Maximize } \mathscr{H}[p_Y(y)] = -\int p_Y(y)\ln[p_Y(y)]dy \\ \text{Subject to } m_Y^{\rho_k} = \int (Y)^{\rho_k}p_Y(y)dy, \quad k = 1, 2, ..., n_k \end{cases} \tag{9}$$

The Lagrangian function associated with Eq. (9) can be written as [19,20,22]

$$\begin{aligned}\mathscr{L}[p_Y(y)] = &-\int p_Y(y)\ln[p_Y(y)]dy - (\lambda_0-1)\left[\int p_Y(y)dy - 1\right] \\ &-\sum_{k=1}^{n_k}\lambda_k\left[\int (Y)^{\rho_k}p_Y(y)dy - m_Y^{\rho_k}\right]\end{aligned} \tag{10}$$

where $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, ..., \lambda_{n_k}]$ denotes the Lagrangian multiplier vector and $\boldsymbol{\rho} = [\rho_1, \rho_2, ..., \rho_{n_k}]$ is the fractional order vector.

For optimal solution, we have

$$\delta\mathscr{L} = \frac{\partial \mathscr{L}}{\partial p_Y(y)} = 0 \tag{11}$$