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Generalized wreath products of table algebras determined by their character tables and applications to association schemes

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ABSTRACT

Structures of table algebras whose character tables have a zero submatrix were studied in Blau and Xu (2014) and Chen and Xu (2018). In this paper we continue the research in this direction, and investigate the conditions under which the character table with a zero submatrix yields a generalized wreath product of table algebras. Applications to association schemes are also discussed.

1. Introduction

The wreath products of table algebras and association schemes have been studied in many papers (for example, see [4,10,13,14] and the references therein). As a generalization of the wreath product, the generalized wreath product is presented in [2,3,5,9], and used to describe the structures of table algebras. The character theory is an important tool in the study of the structures of table algebras and association schemes. In this paper we continue the research in [4,5], and study the conditions under which the character table of a table algebra $(A, \mathbf{B})$ with a zero submatrix yields a generalized wreath product $(\mathbf{B}, \mathbf{N}, \mathbf{T})$, and the condition under which $\mathbf{N}$ is a closed subset of $\mathbf{B}$ (see Theorems 1.3 and 1.5). Applications to association schemes are also discussed. Unlike the character table of a finite group, the character table of a table algebra provides less information about the structure of the table algebra.

In the rest of this introductory section, let us state the main results of the paper explicitly. Let A be a finite dimensional associative algebra over the complex numbers $\mathbb{C}$, with a distinguished basis $\mathbf{B} := \{b_0, b_1, b_2, \ldots, b_d\}$, where $b_0 = 1_A$, the identity element of A. Then $(A, \mathbf{B})$ is called a *table algebra*, and $\mathbf{B}$ is called a *table basis*, if the following properties (i) – (iii) hold:

(i) The structure constants for $\mathbf{B}$ are nonnegative real numbers; that is, $b_ib_j = \sum_{h=0}^{d} \lambda_{ijh}\, b_h$ with $\lambda_{ijh} \in \mathbb{R}_{\geq 0}$, for all $b_i, b_j \in \mathbf{B}$.
(ii) There is an algebra anti-automorphism (denoted by *) of A such that $(a^*)^* = a$ for all $a \in A$ and $b_i^* \in \mathbf{B}$ for all $b_i \in \mathbf{B}$. (Hence we may define i^* by the condition $b_{i^*} := b_i^*$.)
(iii) For all $b_i, b_j \in \mathbf{B}$, $\lambda_{ij0} = 0$ if $j \neq i^*$; and $\lambda_{ii^*0} > 0$.

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Let $(A, \mathbf{B})$ be a table algebra, with $\mathbf{B} := \{b_0 = 1_A, b_1, b_2, \ldots, b_d\}$, and structure constants λ_{ijh}. Then $\lambda_{ii^*0} = \lambda_{i^*i0}$ for all i (cf. the remark after Definition 1.1 in [12]). We may also simply write 1_A as 1 when no ambiguity can occur. It is well known that $(A, \mathbf{B})$ has a unique *positive degree map* χ_0, i.e. an algebra homomorphism $\chi_0 : A \to \mathbb{C}$ such that $\chi_0(b_i) = \chi_0(b_i^*) > 0$ for all $b_i \in \mathbf{B}$ (see [1, Theorem 3.14]). If $\chi_0(b_i) = \lambda_{ii^*0}$, for any $b_i \in \mathbf{B}$, then $(A, \mathbf{B})$ is called a *standard table algebra* (STA), $\mathbf{B}$ a *standard* table basis, and χ_0 the *standard degree map*. If $(A, \mathbf{B})$ is not standard, then the basis $\mathbf{B}$ can be *rescaled* to a standard table basis, replacing each b_i by $\mu_i b_i$, where $\mu_i = \chi_0(b_i)/\lambda_{ii^*0}$. So without loss of generality, in this paper we will always assume that all table algebras are standard. The *order* of any $b_i \in \mathbf{B}$ is $o(b_i) := \lambda_{ii^*0}$. For any $a \in A$ with $a = \sum_{i=0}^{d} \gamma_i b_i$, where $\gamma_i \in \mathbb{C}$, let $\mathrm{Supp}(a) := \{b_i \;:\; \gamma_i \neq 0\}$. The product of nonempty subsets $\mathbf{N}$ and $\mathbf{S}$ of $\mathbf{B}$ is defined by $\mathbf{NS} := \bigcup_{b\in\mathbf{N}, c\in\mathbf{S}} \mathrm{Supp}(bc)$. If $\mathbf{S} = \{c\}$, then we write $\mathbf{NS}$ as $\mathbf{N}c$, and $\mathbf{SN}$ as $c\mathbf{N}$. A nonempty subset $\mathbf{N}$ of $\mathbf{B}$ is called a *closed subset* if $\mathbf{NN} \subseteq \mathbf{N}$. Let $\mathbf{N}$ be a closed subset. Then $b_0 \in \mathbf{N}$, $\mathbf{N}^* = \mathbf{N}$, where $\mathbf{N}^* := \{b^* \;:\; b \in \mathbf{N}\}$, and $(\mathbb{C}\mathbf{N}, \mathbf{N})$ is also a table algebra, where $\mathbb{C}\mathbf{N}$ is the $\mathbb{C}$-space with basis $\mathbf{N}$. Furthermore, we say that $\mathbf{N}$ is *commutative* if $\mathbb{C}\mathbf{N}$ is commutative, and $\mathbf{N}$ is a *normal* closed subset if $b\mathbf{N} = \mathbf{N}b$ for all $b \in \mathbf{B}$.

Definition 1.1 (*Cf. [5, Definition 1.3]*)**.** Let $(A, \mathbf{B})$ be an STA, $\mathbf{T}$ a closed subset of $\mathbf{B}$, and let $\mathbf{N}$ be a subset of $\mathbf{B}$ containing $\mathbf{T}$. If

$$b_i b_j = b_j b_i = o(b_i) b_j, \quad \text{for all } b_i \in \mathbf{T}, b_j \in \mathbf{B}\backslash\mathbf{N},$$

then the triple $(\mathbf{B}, \mathbf{N}, \mathbf{T})$ is called a *generalized wreath product*.

The generalized wreath product is called the partial wreath product in [2]. The definition here is slightly different from that in [2]. In particular, we do not require $\mathbf{N}$ to be a closed subset in the above definition. Let $(\mathbf{B}, \mathbf{N}, \mathbf{T})$ be a generalized wreath product. Then for any closed subset $\mathbf{T}_0 \subset \mathbf{T}$, $(\mathbf{B}, \mathbf{N}, \mathbf{T}_0)$ is also a generalized wreath product. If $\mathbf{T} = \{1_A\}$, then $(\mathbf{B}, \mathbf{N}, \mathbf{T})$ is a *trivial* generalized wreath product. Also note that $(\mathbf{B}, \mathbf{T})$ is a *wreath product* if $(\mathbf{B}, \mathbf{T}, \mathbf{T})$ is a generalized wreath product. The generalized wreath product in table algebras is a generalization of the *Camina triple* in finite groups (cf. [8]).

An irreducible character of $(A, \mathbf{B})$ is also called an irreducible character of $\mathbf{B}$. The set of irreducible characters of $(A, \mathbf{B})$ is denoted by $\mathrm{Irr}(A)$ or $\mathrm{Irr}(\mathbf{B})$. Note that the standard degree map χ_0 is also an irreducible character, called the *principal* irreducible character of A. The *character table* of $(A, \mathbf{B})$, or just the character table of $\mathbf{B}$, is regarded as a matrix whose rows and columns are indexed by the elements of $\mathrm{Irr}(\mathbf{B})$ and $\mathbf{B}$, respectively, and whose (χ, b)-entry is $\chi(b)$, for any $\chi \in \mathrm{Irr}(\mathbf{B})$ and $b \in \mathbf{B}$. The rows and columns of the character table of $\mathbf{B}$ can be arranged in any order. Let $\mathbf{N}$ be a nonempty subset of $\mathbf{B}$, and Φ a nonempty subset of $\mathrm{Irr}(\mathbf{B})$. Then we can write the character table as a block matrix of the following form:

$$\begin{array}{cc} & \mathbf{N} \\ \Phi & \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \end{array}$$

where the columns below $\mathbf{N}$ are indexed by the elements of $\mathbf{N}$, and the rows to the right of Φ are indexed by the elements of Φ. For any positive integers s and t, the $s \times t$ zero matrix is denoted by $O_{s\times t}$. In this paper we are interested in the table algebras whose character tables can be written as a block matrix of the form

$$\begin{array}{cc} & \mathbf{N} \\ \Phi & \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & O_{s\times t} \end{pmatrix}. \end{array} \tag{1.1}$$

Assume that the character table of $(A, \mathbf{B})$ is of the form in (1.1). Then $s+t \le |\mathbf{B}|-1$ by [4, Proposition 3.1]. If $s+t = |\mathbf{B}|-1$, the next theorem describes the structure of $(A, \mathbf{B})$ in terms of wreath product. The *degree* of $\chi \in \mathrm{Irr}(\mathbf{B})$ is $\chi(1)$, and χ is said to be *linear* if $\chi(1) = 1$.

Theorem 1.2 (*Cf. [4, Theorem 1.8]*)**.** *Let $(A, \mathbf{B})$ be an STA. Then the following are equivalent.*

(i) *There is a proper commutative closed subset $\mathbf{N}$ of $\mathbf{B}$ such that $(A, \mathbf{B})$ is a wreath product $(\mathbf{B}, \mathbf{N})$.*
(ii) *By permuting the rows and columns if necessary, the character table of $(A, \mathbf{B})$ has the form in (1.1) such that $s+t = |\mathbf{B}|-1$, and every $\chi \in \mathrm{Irr}(\mathbf{B})\backslash\Phi$ is linear.*

A natural question is how to describe the structure of a table algebra whose character table is of the form in (1.1) with $s+t = |\mathbf{B}|-2$. It is the purpose of this paper to give an answer to this question.

Let $(A, \mathbf{B})$ be an STA. For any nonempty subset $\mathbf{N}$ of $\mathbf{B}$, let $o(\mathbf{N}) := \sum_{b\in\mathbf{N}} o(b)$, and $\mathbf{N}^+ := \sum_{b\in\mathbf{N}} b$. Given a closed subset $\mathbf{T}$ of $\mathbf{B}$, let $b/\!\!/\mathbf{T} := o(\mathbf{T})^{-1}(\mathbf{T}b\mathbf{T})^+$, for any $b \in \mathbf{B}$, and $\mathbf{B}/\!\!/\mathbf{T} := \{b/\!\!/\mathbf{T} \;:\; b \in \mathbf{B}\}$. Let $A/\!\!/\mathbf{T}$ be the $\mathbb{C}$-space with basis $\mathbf{B}/\!\!/\mathbf{T}$. Then $(A/\!\!/\mathbf{T}, \mathbf{B}/\!\!/\mathbf{T})$ is also an STA, called the *quotient* table algebra of $(A, \mathbf{B})$ with respect to $\mathbf{T}$. For any nonempty subset $\mathbf{N}$ of $\mathbf{B}$, let $\mathbf{N}/\!\!/\mathbf{T} := \{b/\!\!/\mathbf{T} \;:\; b \in \mathbf{N}\}$.

Let Φ be a nonempty subset of $\mathrm{Irr}(\mathbf{B})$. As in [5], define a binary relation $\sim_\Phi$ on $\mathbf{B}$ as follows:

$$b_i \sim_\Phi b_j \qquad \text{if and only if} \qquad \chi(b_i)/o(b_i) = \chi(b_j)/o(b_j), \ \text{for all } \chi \in \Phi.$$

It is clear that $\sim_\Phi$ is an equivalence relation. An equivalence class of $\sim_\Phi$ is simply called a $\sim_\Phi$*-class*. The *kernel* of $\chi \in \mathrm{Irr}(\mathbf{B})$ is $\ker\chi := \{b \in \mathbf{B} \;:\; \chi(b) = o(b)\chi(1)\}$, and $\ker\chi$ is a closed subset of $\mathbf{B}$ (cf. [11, Theorem 4.2]).