Discrete Mathematics 342 (2019) 96–112

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# A class of highly symmetric graphs, symmetric cylindrical constructions and their spectra

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ARTICLE INFO

*Article history:*
Received 13 October 2017
Received in revised form 14 August 2018
Accepted 15 September 2018
Available online xxxx

*Keywords:*
Graph spectra
Graph products
Graph amalgam
Cylindrical construction
Graph decomposition

ABSTRACT

In this article, we introduce the algebra of block-symmetric cylinders and we show that symmetric cylindrical constructions on base-graphs admitting commutative decompositions behave as generalized tensor products. We compute the characteristic polynomial of such symmetric cylindrical constructions in terms of the spectra of the base-graph and the cylinders in a general setting. This gives rise to a simultaneous generalization of some well-known results on the spectra of a variety of graph amalgams, as various graph products, graph subdivisions and generalized Petersen graph constructions. While our main result introduces a connection between spectral graph theory and commutative decompositions of graphs, we focus on commutative cyclic decompositions of complete graphs and tree-cylinders along with a subtle group labeling of trees to introduce a class of highly symmetric graphs containing the Petersen and the Coxeter graphs. Also, using techniques based on recursive polynomials we compute the characteristic polynomials of these highly symmetric graphs as an application of our main result.


## 1. Introduction

### 1.1. Background and main results

Nowadays, *graph amalgams* is a classic chapter in graph theory while the study of graph products and their spectra goes back to the early days of this field of study [11] (also see [3,5,12]). The cylindrical graph construction as a generalized edge-replacement procedure, introduced in [6] (also see [17]), is a graph amalgam that unifies a large number of graph constructions including different kinds of graph products while it also introduces some new interesting amalgams. On the other hand, study of graphs through assigning matrices that encode structural properties is also an old section of graph theory with a large intersection with discrete spectral geometry and its methods. These along with the main duality proved in [6] lead to the following basic question:

"How far the cylindrical amalgam generalizes the tensor product of matrices and its properties?"

In Section 2 we consider a subclass of cylindrical constructs that can be considered as generalized tensor products. Our basic goal in this section is to introduce a general result (Theorem 1) that describes the spectrum of the construct in terms of the spectra of its components where commutativity conditions are imposed. In this regard, we introduce the concept of a *commutative t-decomposition* where in our main theorem the base-graph is decomposed into a commutative set of spanning subgraphs by the corresponding labeling. We not only deduce a number of known results as corollaries of our main theorem

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https://doi.org/10.1016/j.disc.2018.09.022

(see Examples 2 and 3 and compare to [1,4,8,20]), but also, in our opinion, another interesting aspect of Theorem 1 is to introduce a connection between the theory of graph decompositions and the theory of graph spectra (e.g. see Example 2).

On the other hand, investigating symmetries of graphs and their relations to multiplicities of eigenvalues is also an old topic in spectral graph theory (e.g. see [10] and references therein), while constructing sparse highly symmetric graphs is among the most fundamental problems in graph theory with deep applications in mathematics and computer science (e.g. see [21] and Section 4). In this regard, in Section 3 as another application of our main result, using the concept of tree-cylinders introduced in [17], we consider a special family of cylindrical constructions in which complete graphs are sparsified by regular tree-cylinders. Moreover, we analyze the spectra of such constructions in detail. Hence, easily follows that such a study is closely related to the study of the spectra of regular trees in which leaves are connected to each other in a predefined way related to the twists of the construction. In particular, we distinguish a special highly symmetric case of this construction that can be viewed as a generalization of the Petersen and the Coxeter graphs. We believe that these graphs deserve more investigation in future research (see Section 4 for a discussion on this).

Summing up, the main result of the paper is Theorem 1 presented in Section 2.2, and called the main spectral theorem. Among others, it shows that if the considered base-graph is decomposed into a commutative set of spanning subgraphs by the corresponding labeling then the cylindrical construction by block-symmetric cylinders behaves as a generalized tensor product, and consequently, the corresponding characteristic polynomial is computed using the main spectral theorem. This links the fundamental subjects, graph decompositions and graph spectra in a general setup, making it possible to use results on graph decompositions to construct graphs with prescribed properties for their spectra. Also, some important consequences of Theorem 1 are the results of Section 3, and in particular Proposition 2 showing that the main spectral theorem can be used to compute the spectra of sparsifications by tree-cylinders (see [6]), and in particular, such sparsifications of complete graphs, giving rise to a class of highly symmetric graphs discussed in Section 3.1. Further discussions on our results will appear in Section 4.

### 1.2. Notations and basic concepts

To add a couple of words on notations, we note that hereafter we only deal with finite loopless undirected multigraphs which are referred to using math-Roman font as H, where matrices are referred to by italic-Roman font as $H$. Note that in this setting, the adjacency matrix of such multigraph is defined to be the adjacency matrix of the corresponding weighted graph in which the integer weight of each edge is the number of multiedges in the original graph. In the sequel, $\mathrm{K}_n$ is the complete graph on $n$ vertices and $\mathrm{P}_n$ is the path of length $n-1$. A regular tree is a tree whose non-leaf vertices have the same degree.

Throughout this paper we denote by $\mathbb{R}$ the real number field, and by a matrix we mean a matrix with real entries in $\mathbb{R}$ unless it is stated otherwise. The symbols $I$ and $J$ stand for the identity matrix and the all-one matrix, respectively, and we define

$$\bar{I} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{0} & & 1 \\ & \iddots & \\ 1 & & \mathbf{0} \end{bmatrix}$$

where the dimensions of these matrices are assumed to be clear from the context. The transpose of a matrix $H$ is denoted by $H^T$ and the characteristic polynomial of a matrix $H$ in terms of the variable $x$ is defined as $\phi(H, x) \stackrel{\text{def}}{=} \det(xI - H)$.

A key role in this paper is playing by the following useful concept.

**Definition 1.** A square $2 \times 2$ block matrix

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

is defined to be *block-symmetric* (or *bl-symmetric* for short), if all blocks are symmetric matrices, $A_{1,1} = A_{2,2}$, and $A_{1,2} = A_{2,1}$.

Hereafter, given an integer $k \geq 1$, we denote by $\mathcal{B}_k \stackrel{\text{def}}{=} \mathcal{B}_k(\mathbb{R})$ the $\mathbb{R}$-algebra of all block square $2 \times 2$ matrices of size $k$ with entries in $\mathbb{R}$, and by $\mathcal{BS}_k \stackrel{\text{def}}{=} \mathcal{BS}_k(\mathbb{R})$ the $\mathbb{R}$-subalgebra of $\mathcal{B}_k$ formed by all bl-symmetric matrices. Also, in this setting $\mathcal{BS}$ stands for the whole class of bl-symmetric matrices. ▲

Moreover, let us recall a standard determinant identity for block matrices in $\mathcal{B}_k$ as follows (e.g. see [23]):

$$\det \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} = \det(A_{1,1} - A_{1,2}A_{2,2}^{-1}A_{2,1}) \det(A_{2,2}), \tag{1}$$

whenever $A_{2,2}$ is an invertible matrix.