# The fiber dimension of a graph 

Tobias Windisch<br>Otto-von-Guericke Universität, Magdeburg, Germany

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#### Abstract

Graphs on integer points of polytopes whose edges come from a set of allowed differences are studied. It is shown that any simple graph can be embedded in that way. The minimal dimension of such a representation is the fiber dimension of the given graph. The fiber dimension is determined for various classes of graphs and an upper bound in terms of the chromatic number is proven.


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## 1. Introduction

The study of geometric properties of graphs is a key ingredient in understanding their algorithmic behavior and combinatorial structure [33,26]. In [16], the dimension of a graph was introduced as the smallest $n \in \mathbb{N}$ such that the graph can be embedded in $\mathbb{R}^{n}$ with every edge having unit length. Recently, isometric embeddings of graphs into discrete objects like hypercubes or lattices have received a lot of attention in the literature and lead to many variations of graph dimension, like the isometric dimension [17], the lattice dimension [15,23], or the Fibonacci dimension [9]. In this paper, a new notion is added to the list of graph dimensions. Our concept has its origin in algebraic statistics, an emerging field which explores statistical questions with algebraic tools $[13,14,19,4]$. A main task there is to construct connected graphs on integer points of polytopes in order to draw samples by performing a random walk [32, Chapter 5]. For a given polytope $P \subset \mathbb{Q}^{d}$ and a symmetric set $\mathcal{M} \subset \mathbb{Z}^{d}$, a graph on $P \cap \mathbb{Z}^{d}$ is given by connecting two nodes $u$ and $v$ by an edge if $u-v \in \mathcal{M}$. Graphs which can be obtained in that way are often referred to as fiber graphs in the literature and can be understood as a discrete analogue of unit distance graphs [8].

As every graph $G$ can be represented as a fiber graph (Proposition 2.3), this motivates the question for the smallest dimension in which such a representation exists, the fiber dimension of $G$ (Definition 2.6). We explore general properties of this dimension and state upper bounds in terms of the number of nodes (Theorem 3.10) and the chromatic number (Theorem 3.5) in the spirit of [16]. We then determine the fiber dimension for a variety of graphs. The fiber dimension of a cycle of length $n$ depends on Euler's totient function and we show that $f \operatorname{dim}\left(C_{n}\right)=1$ if and only if $n \in \mathbb{N} \backslash\{3,4,6\}$. Cycles whose length is one of the exceptional cases $n \in\{3,4,6\}$ have fiber dimension two (Proposition 4.7). Its proof uses the well-known fact that Euler's totient function of $n \in \mathbb{N}$ is two if and only if $n \in\{3,4,6\}$. We also determine the fiber dimension of complete graphs and show that it is logarithmic in the number of nodes (Theorem 5.4). A connection to distinct pair-sum polytopes [10] is established and it is shown how the fiber dimension leads to relations between the number of lattice points and the dimension of the ambient space of these polytopes. In Section 6, we present methods to decide whether the fiber

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Fig. 1. A fiber graph in $\mathbb{Q}^{2}$.
dimension of a graph is smaller or equal to three. Eventually, we discuss the obstacles that make the computation of the fiber dimension challenging in higher dimensions.
Conventions and notations. The natural numbers are denoted by $\mathbb{N}=\{0,1,2, \ldots\}$ and for $n \in \mathbb{N}$ with $n>0$, $[n]:=\{1,2, \ldots, n\}$. All graphs that appear in this paper are simple, i.e., all edges are undirected and they do not have loops or multiple edges. For any graph $G$, the set of nodes is denoted by $V(G)$ and its chromatic number is $\chi(G)$. The unit vectors of $\mathbb{Q}^{d}$ are denoted by $e_{1}, \ldots, e_{d}$. For any $n \in \mathbb{N}, K_{n}$ and $C_{n}$ denote the complete graph and the cycle graph on $n$ nodes respectively.

## 2. Fiber graphs

A polytope $P \subset \mathbb{Q}^{d}$ is a lattice polytope if all its vertices are in $\mathbb{Z}^{d}$. A finite set $\mathcal{M} \subset \mathbb{Z}^{d} \backslash\{0\}$ is a set of moves if $\mathcal{M}=-\mathcal{M}$ and if for all $\lambda \in \mathbb{N}$ with $\lambda \geq 2$ and all $m \in \mathcal{M}, \lambda \cdot m \notin \mathcal{M}$.

Definition 2.1. Let $P \subset \mathbb{Q}^{d}$ be a lattice polytope and let $\mathcal{M} \subset \mathbb{Z}^{d}$ be a set of moves. The fiber graph $P(\mathcal{M})$ is the graph on $P \cap \mathbb{Z}^{d}$ where two nodes $v$ and $u$ are adjacent if $u-v \in \mathcal{M}$.

In applications, fiber graphs (see Fig. 1) arise typically from graphs on fibers of matrices $A \in \mathbb{Q}^{m \times d}$, that are sets of the form $A^{-1} b:=\left\{u \in \mathbb{N}^{d}: A u=b\right\}$ for $b \in \mathbb{Q}^{m}$. These sets appear naturally in integer programming, where the goal is to find an optimal solution over the fiber with respect to some objective function. Another example where fiber graphs appear is from hypothesis testing in statistics, where one wants to verify how good the log-linear model defined by $A$ explains observed data encoded in $b$ (see for instance [1, Chapter 8] and references therein). Both problem statements have in common that they require to explore a discrete set which is given implicitly by linear equalities. Fiber graphs provide an algorithmic framework for the exploration of $A^{-1} b$, either randomly [13] or systematically [ 12 , Chapter 2 ], as the finite set of moves allows to enumerate the neighborhood of any node. This property suffices to perform random walks on these types of graphs and properties of the underlying graph influence the convergence behavior of the random walk [22,25]. In integer programming one needs to put additional conditions on the set of allowed moves. For instance, a set of moves which has proven to be efficient in finding an integer solution [20] is the Graver basis of $A$ [18]. A particular situation is when the set of moves makes the fiber graph a connected graph:

Definition 2.2. Let $P \subset \mathbb{Q}^{d}$ be a lattice polytope. A set of moves $\mathcal{M} \subset \mathbb{Z}^{d}$ is a Markov basis of $P$ if the fiber graph $P(\mathcal{M})$ is connected.

The notion Markov bases comes from algebraic statistics [14], where one wants to run irreducible Markov chains on fibers and hence finding a set of moves that connects the fiber is a key challenge. With tools from commutative algebra [13,14], a universal set of moves $\mathcal{M} \subset \operatorname{ker}_{\mathbb{Z}}(A)$ can be computed such that the fiber graphs on all fibers of $A$ are connected simultaneously. For more on fiber graphs in applications we recommend the excellent textbooks [12,32,14].

On the first look, it may seem that fiber graphs form a special class of graphs on their own, but, as the next proposition shows, every graph is a fiber graph:

Proposition 2.3. Every simple graph is isomorphic to a fiber graph.
Proof. Let $G=\left(\left\{v_{1}, \ldots, v_{n}\right\}, E\right)$ be a graph and let $P:=\left\{x \in \mathbb{Q}_{\geq 0}^{n}: \sum_{i=1}^{n} x_{i}=1\right\}$ be the $(n-1)$-dimensional simplex, then $P \cap \mathbb{Z}^{n}=\left\{e_{1}, \ldots, e_{n}\right\}$. Consider $\mathcal{M}:=\left\{e_{i}-e_{j}:\left\{v_{i}, v_{j}\right\} \in E\right\}$, then $P(\mathcal{M})$ is isomorphic to $G$.

The next lemma states that every fiber graph can be written as a fiber graph in a full dimensional polytope.
Lemma 2.4. Let $P \subset \mathbb{Q}^{m}$ be a d-dimensional polytope and $\mathcal{M} \subset \mathbb{Z}^{m}$ a set of moves. There exists ad-dimensional polytope $P^{\prime} \subset \mathbb{Q}^{d}$ and a set of moves $\mathcal{M}^{\prime} \subset \mathbb{Z}^{d}$ so that $P(\mathcal{M}) \cong P^{\prime}\left(\mathcal{M}^{\prime}\right)$.

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[^0]:    E-mail address: tobias.windisch@posteo.de.

