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Asymptotic analysis of capillary–gravity waves generated by a moving disturbance

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ABSTRACT

Capillary–gravity waves generated by a steadily translating disturbance are studied via a direct analysis based on the geometrical relationship between dispersion curves on the Fourier plane and the corresponding wave pattern on the free surface, through which wave crestlines, wavelength, cusp angles and phase & group velocities are explicitly obtained.

1. Introduction

When a body (such as: a ship with large scale or an insect with small scale) or a pressure patch travels along a straight path at a constant speed c on the air–water interface, free-surface waves, in the form of a stationary wave pattern, are generated. In the analysis of such stationary waves, the surface tension effect is usually ignored inasmuch as the surface tension effect is predominant only for very short waves with the wavelength in the order of centimeters [1,2]. Since Lord Kelvin [3], the stationary waves generated by a moving obstacle have been studied extensively. In deepwater, its pattern composed of transverse and divergent wave systems is confined in a V-shaped region (classical Kelvin's ship wave pattern), and the half-angle of this sector is called as "Kelvin angle" which is $\Gamma_K = \arcsin 1/3 \approx 19.47^\circ$ independent of the ship's speed [4,5]. The Kelvin angle can be expounded by performing stationary phase analysis of the dispersion relation [5,6] or analyzing the propagating direction of the group velocity [7].

Interestingly, the free-surface wave pattern created by a stead ily moving obstacle can be totally different from the Kelvin's ship wave pattern when the surface tension effect is accounted for [8–10]. The case of combined effects of capillarity and gravity on free-surface waves generated by a translating object has been considered recently [9–13]. Both experimental observation and numerical calculation show that there are waves in front of the disturbance [10–12], which cannot be predicted by the theory of pure-gravity waves. In addition, one notable behavior is that there is no wave generated and the corresponding wave resistance is zero when the traveling speed is less than the minimum speed of capillary–gravity waves $c_{\min} \approx 0.2313$ m/s [4,5].

Besides, in the classical hydrodynamic textbook by Lamb [4], two typical wave patterns are presented. At a low forward speed, both gravity and capillarity effects are important, and the wave pattern is smooth and curved without divergent waves. However, at a high speed, the surface tension effect is insignificant, and the wave pattern is close to the Kelvin's ship wave pattern. It is estimated that the disappearance of divergent waves occurs at $c \approx 2c_{\min}$ [14]. The exact transition speed was first found by Binnie [15] which is approximately equal to $c \approx 1.938c_{\min}$, and then wave patterns crossing the transition speed are described in [10,9,16,13].

In the present paper, we study the stationary waves with the combined capillary–gravity effect associated with a point perturbation based on a direct analysis via establishing the geometrical relationship between dispersion curves on the Fourier plane and the corresponding wave pattern on the free surface. This relationship was firstly established by Crapper [14]. This technique was also applied to analyze the unsteady ship waves as well as steady ship waves in finite water depth [17,18]. By using this relationship, far-field wave profiles, cusp angles and phase & group velocities can be determined in an explicit and simple way. Through investigating the inflection points along the dispersion curve, the wave systems are classified. At $c_{\text{div}} \approx 0.4484$ m/s which is consistent with the finding in [15], two inflection points coincide and divergent waves disappear. Besides, another speed $c_D \approx 0.6800$ m/s

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associated with the angle of demarcation asymptote of gravity-dominant and capillarity-dominant waves, which has never been reported in the literature, is obtained. Behaviors of the wave pattern and cusp angles for different velocities across critical speeds c_{div} and c_D are investigated.

The layout of the present paper is organized as following. In Section 2, the dispersion relation associated with the linear free-surface boundary condition is outlined, and dispersion curves on the Fourier plane are discussed. In Section 3, the relationship between dispersion curves on the Fourier plane and the free-surface wave pattern in the physical space is established, and cusp angles associated with inflection points along the dispersion curves are studied. In Section 4, features of dispersion curve and the corresponding far-field wave pattern for traveling speeds c across the critical speeds c_{div} and c_D are exhibited. Cusp angles associated with inflection points and angle of demarcation are set forth. Finally, concluding remarks are presented in Section 5.

2. Dispersion relation and dispersion curves

Consider an incompressible, inviscid water domain with infinite depth and lateral extent bounded on the top by an air–water interface free surface. A three-dimensional Cartesian coordinate system $OXYZ$ steadily traveling with the disturbance at a constant speed c in the direction of positive OX axis is defined with the XY plane coinciding with the undisturbed free surface and the OZ axis orienting positively upwards. Here, the gravitational acceleration $g = 9.8067$ m/s^2, translating speed c and water density $\rho = 1000$ kg/m^3 are used to define nondimensional coordinates (x, y, z), Fourier variables (α, β, k), velocity potential ϕ and pressure p as:

$$(x, y, z) = \frac{g}{c^2}(X, Y, Z), \qquad (\alpha, \beta) = \frac{c^2}{g}(A, B), \qquad \phi = \frac{g}{c^3}\Phi, \qquad p = \frac{P}{\rho c^2}. \tag{1}$$

The characteristic wavenumber of capillary waves is $K_T = \sqrt{\rho g/T}$ [5] while the fundamental wavenumber of pure-gravity stationary waves is $K_G = g/c^2$, so that the parameter σ associated with the surface tension effect is defined as the ratio of both [14,8]:

$$\sigma = \frac{K_G}{K_T} = \frac{g}{c^2}\sqrt{\frac{T}{\rho g}}, \tag{2}$$

where T represents the air–water interface tension $T \approx 0.073$ N/m [4,5]. The parameter σ represents the significance of the surface tension effect. From (2), the surface tension effect is important at a low forward speed, whereas it can be ignored when the translation speed is high. Then, the governing equation and linear free-surface boundary conditions in nondimensional forms are written as:

$$\nabla^2\phi = 0 \quad \text{in the fluid domain}, \tag{3a}$$

$$p - \frac{\partial\phi}{\partial x} + \mathcal{E} + \sigma^2\left(\frac{\partial^2\mathcal{E}}{\partial x^2} + \frac{\partial^2\mathcal{E}}{\partial y^2}\right) = 0 \quad \text{on} \quad z = 0, \tag{3b}$$

$$-\frac{\partial\mathcal{E}}{\partial x} = \frac{\partial\phi}{\partial z} \quad \text{on} \quad z = 0, \tag{3c}$$

where $\mathcal{E}$ means the free-surface elevation. Introducing the Fourier transform with respect to physical quantities p, ϕ and $\mathcal{E}$ yields [19]:

$$\begin{Bmatrix}\hat{p}\\ \hat{\phi}\\ \hat{\mathcal{E}}\end{Bmatrix} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{Bmatrix}p\\ \phi\\ \mathcal{E}\end{Bmatrix} e^{i(\alpha x+\beta y)}dxdy, \tag{4}$$

By performing the Fourier transform to the Laplace equation and free-surface boundary conditions given by (3) as conducted in [5], we can obtain:

$$\hat{\mathcal{E}} = \frac{\sqrt{\alpha^2+\beta^2}\,\hat{p}}{\alpha^2 - \sqrt{\alpha^2+\beta^2} - \sigma^2\left(\alpha^2+\beta^2\right)^{3/2}}, \tag{5}$$

then the expression of the free-surface elevation can be obtained:

$$\mathcal{E}(x, y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{p}(\alpha, \beta)\,\frac{\sqrt{\alpha^2+\beta^2}}{D(\alpha, \beta)}e^{-i(\alpha x+\beta y)}d\alpha d\beta, \tag{6}$$

where the function $D(\alpha, \beta)$ in the denominator represents the dispersion function defined on the (α, β) Fourier plane [5]:

$$D(\alpha, \beta) = \alpha^2 - \sqrt{\alpha^2+\beta^2} - \sigma^2\left(\alpha^2+\beta^2\right)^{3/2}, \tag{7}$$

where the wavenumber vector is defined as: $(\alpha, \beta) = k(\cos\theta, \sin\theta)$. By enforcing $D(\alpha, \beta) = 0$, we obtain the dispersion relationship associated with the linear free-surface boundary condition. By plotting $D = 0$ on the (α, β) Fourier plane, we get dispersion curves. Dispersion curves defined by $D = 0$ are symmetrical with respect to both axes $\alpha = 0$ and $\beta = 0$. In the quadrant $\alpha \geq 0$ and $\beta \geq 0$, dispersion curves are defined explicitly in polar Fourier coordinates (k, θ) by two characteristic wavenumbers:

$$\begin{cases} k_G = \dfrac{2}{\cos^2\theta + \sqrt{\cos^4\theta - 4\sigma^2}} & \text{as} \quad k \leq k_D, \\[2ex] k_T = \dfrac{\cos^2\theta + \sqrt{\cos^4\theta - 4\sigma^2}}{2\sigma^2} & \text{as} \quad k \geq k_D, \end{cases} \tag{8}$$

where $k_D = 1/\sigma$ denotes the wavenumber dividing the dispersion curve into gravity-dominant and capillarity-dominant components. In Eq. (8), $k_G \leq k_D$ denotes the wavenumber of gravity-dominant waves, and $k_T \geq k_D$ represents the wavenumber of waves where the capillarity plays a dominant role [14]. The dispersion curve is closed and confined in the following region:

$$0 \leq |\theta| \leq \theta_D = \arctan\sqrt{\frac{1-2\sigma}{2\sigma}}, \tag{9}$$

When $\sigma > 1/2$, θ_D does not exist, nor does the dispersion curve. At the critical value $\sigma = 1/2$ with the corresponding speed defined as $c_{\min} \approx 0.2313$ m/s, the dispersion curve reduces to an isolated point at $(\alpha, \beta) = (2, 0)$ and $\theta_D = 0$ according to expression (9) which means that all waves disappear since the obstacle's traveling speed is less than the minimum speed of capillary–gravity waves. Therefore, free-surface waves cannot be generated when the traveling speed c is less than the critical speed $c_{\min} \approx 0.2313$ m/s. At $\theta = \theta_D$, we have $k = k_D = \sqrt{\alpha_D^2 + \beta_D^2}$ as displayed in Fig. 1. At $\theta = 0$, we define two wavenumbers k_G^0 and k_T^0 as:

$$k_G^0 = \frac{2}{1+\sqrt{1-4\sigma^2}} \quad \text{and} \quad k_T^0 = \frac{1+\sqrt{1-4\sigma^2}}{2\sigma^2}, \tag{10}$$

so that the dispersion curve intersects the α-axis at a minimum wavenumber $\alpha = k_G^0$ and a maximum wavenumber $\alpha = k_T^0$ as depicted in Fig. 1. Wavenumbers k_G^0 and k_T^0 are associated with the wavelengths along the translating track of the disturbance. The wavelengths of gravity-dominant and capillarity-dominant waves are given by

$$\lambda_G^0 = \frac{2\pi}{k_G^0} = \pi\left(1+\sqrt{1-4\sigma^2}\right) \quad \text{and} \quad \lambda_T^0 = \frac{2\pi}{k_T^0} = \frac{4\pi\sigma^2}{1+\sqrt{1-4\sigma^2}}. \tag{11}$$

Dispersion curves expressed in (8) on the (α, β) Fourier plane are displayed in Fig. 1 for $\sigma = 0$ when the surface tension effect is ignored and $\sigma = 0.275$ when the surface tension effect is accounted for. The dispersion curve without the surface tension