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Brief paper

On the converse of the passivity and small-gain theorems for input–output maps[☆]

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ABSTRACT

We prove the following converse of the passivity theorem. Consider a causal system given by a sum of a linear time-invariant and a passive linear time-varying input–output map. Then, in order to guarantee stability (in the sense of finite $\mathbf{L}_2$-gain) of the feedback interconnection of the system with an *arbitrary* nonlinear output strictly passive system, the given system must itself be output strictly passive. The proof is based on the S-procedure lossless theorem. We discuss the importance of this result for the control of systems interacting with an output strictly passive, but otherwise completely unknown, environment. Similarly, we prove the necessity of the small-gain condition for closed-loop stability of certain time-varying systems, extending the well-known necessity result in linear robust control.

1. Introduction

The *passivity* and *small-gain* theorems are fundamental to large parts of systems and control theory, see e.g. Megretski and Rantzer (1997), Moylan and Hill (1978), van der Schaft (2017), Vidyasagar (1981) and Willems (1972). Both theorems provide a stability 'certificate' when feedback interconnecting the given system with an arbitrary system which is either (in the small-gain setting) assumed to have an L_2-gain smaller than the reciprocal of the L_2-gain of the given system, or is (output strictly) passive like the given system. These theorems are valid from linear finite-dimensional systems to nonlinear and infinite-dimensional systems.

The current paper is concerned with the *converse* of these theorems; that is the *necessity* of the (strict) passivity or the small-gain condition for closed-loop stability when interconnecting in feedback a given system with an *arbitrary* system, which is *unknown* apart from a passivity or $\mathbf{L}_2$-gain assumption. Surprisingly, this converse of the *passivity* theorem has hardly been studied in the literature; despite its fundamental importance in applications. For example, in order to guarantee stability of a controlled robotic system interacting with a passive, but else completely *unknown*, environment, the converse of the passivity theorem tells us that the controlled robot *must* be output strictly passive as seen from the interaction port of the robot with the environment. This has far-reaching methodological implications for control design, since it means that rendering by control the system output strictly passive at the interaction port is not only a valid option, but is also the *only* option guaranteeing stability for an unknown passive environment. The same holds within the context of robust nonlinear control whenever we replace 'environment' by the uncertain part of the system.

Up to now this converse passivity theorem was only proved for *linear time-invariant single-input single-output* systems in Colgate and Hogan (1988), using arguments from Nyquist stability theory,[1] exactly with the robotics motivation in mind. The same motivation was elaborated on in Stramigioli (2015), where the following form of a converse passivity theorem was obtained for nonlinear systems in state space form. If a system is *not* passive then for any given constant K one can define a passive system that extracts from the given system an amount of energy that is larger than K, implying that the norm of the state of the constructed system becomes larger than K, thereby demonstrating some sort of instability of the closed-loop system. In the present paper, a converse of the

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[1] Roughly speaking, by showing that if Σ_1 is *not* passive, a positive-real transfer function (corresponding to a passive system Σ_2) can be constructed such that the closed-loop system fails the Nyquist stability test.

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passivity theorem will be derived for a class of *input–output maps*, namely those decomposable into a sum of a linear time-invariant map and a passive linear time-varying map. This converse passivity theorem involves feedback interconnections with nonlinear systems and will be formulated in three versions in Section 3, with their own range of applicability. In all cases the proofs are based on the S-procedure lossless theorem due to Megretski and Treil (1993); see also Jönsson (2001, Thm. 7).

Converse statements of the *small-gain* theorem are much more present in the literature; see e.g. Zhou, Doyle, and Glover (1996, Theorem 9.1) for the finite-dimensional linear case and Curtain and Zwart (1995) for infinite-dimensional linear systems. However, to the best of our knowledge, the converse of the small-gain theorem for *linear time-varying* systems interconnected in feedback with nonlinear systems, as obtained in Section 4, is new, while also the proof line is different from the existing one. Similarly to the passivity case, this converse will be formulated for a class of linear time-varying input–output maps, and the proofs, in two different versions, will be based on the S-procedure lossless theorem.

Finally, Section 5 presents the conclusions, and discusses problems for further research. A preliminary version of some of the results in Section 3 of this paper was presented at the IFAC World Congress 2017; cf. Khong and van der Schaft (2017).

2. Preliminaries

This section summarizes the background for this paper; see e.g. van der Schaft (2017) for details. Denote the set of $\mathbb{R}^n$-valued Lebesgue square-integrable functions by

$$\mathbf{L}_2^n := \left\{ v : [0,\infty) \to \mathbb{R}^n \mid \|v\|_2^2 := \int_0^\infty v(t)^T v(t)\,dt < \infty \right\}.$$

For any two $v, w \in \mathbf{L}_2^n$ denote the $\mathbf{L}_2^n$-inner product as

$$\langle v, w\rangle := \int_0^\infty v(t)^T w(t)\,dt$$

Define the *truncation* operator $(P_T v)(t) := v(t)$ for $t \leq T$; $(P_T v)(t) := 0$ for $t > T$, and the extended function space

$$\mathbf{L}_{2e}^n := \{ v : [0,\infty) \to \mathbb{R}^n \mid P_T v \in \mathbf{L}_2,\ \forall T \in [0,\infty)\}.$$

In what follows, the superscript n will often be suppressed for notational simplicity. Throughout this paper a *system* will be specified by an input–output map $\Delta : \mathbf{L}_{2e}^m \to \mathbf{L}_{2e}^p$ satisfying $\Delta(0) = 0$.

Define for any $\tau \geq 0$ the *right shift* operator $(S_\tau(u))(t) = u(t-\tau)$ for $t \geq \tau$ and $(S_\tau(u))(t) = 0$ for $0 \leq t < \tau$. The system Δ is said to be *time-invariant* if $S_\tau \Delta = \Delta S_\tau$ for all $\tau > 0$. Furthermore, the system Δ is *bounded* if Δ maps $\mathbf{L}_2^m$ into $\mathbf{L}_2^p$. It is said to have $\mathbf{L}_2$-*gain* $\leq \gamma$ for some $\gamma > 0$ (*finite* $\mathbf{L}_2$-gain) if

$$\|P_T \Delta(u)\|_2 \leq \gamma \|P_T u\|_2 \tag{1}$$

for all $u \in \mathbf{L}_{2e}^m$ and all $T \geq 0$. The infimum of all γ satisfying (1) is called the $\mathbf{L}_2$-gain of Δ. The system Δ is *causal* if $P_T \Delta P_T = P_T \Delta$ for all $T \geq 0$. It is well-known, see e.g. van der Schaft (2017, Proposition 1.2.3), that a causal system Δ has finite $\mathbf{L}_2$-gain if and only if, instead of (1),

$$\|\Delta(u)\|_2 \leq \gamma \|u\|_2 \tag{2}$$

for all $u \in \mathbf{L}_2^m$. For the purpose of *interconnection* of systems the above notions are generalized from maps to *relations* $R \subset \mathbf{L}_{2e}^m \times \mathbf{L}_{2e}^p$ satisfying $(0,0) \in R$ as follows van der Schaft (2017). A relation R is said to be *bounded* if whenever $(u,y) \in R$ and $u \in \mathbf{L}_2$ then also $y \in \mathbf{L}_2$. Furthermore, R has *finite* $\mathbf{L}_2$-*gain* if

$$\|P_T y\|_2 \leq \gamma \|P_T u\|_2 \tag{3}$$

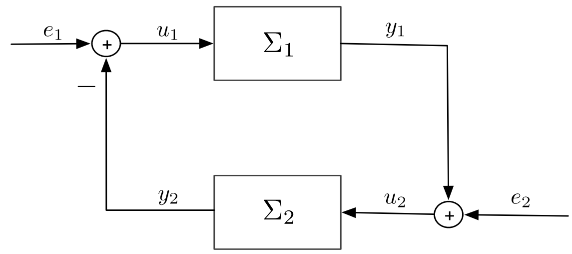

Fig. 1. Feedback configuration.

for all $(u,y) \in R$ and all $T \geq 0$. Also, R is said to be *causal* if whenever $(u_1,y_1) \in R$, $(u_2,y_2) \in R$ satisfy $P_T u_1 = P_T u_2$, then $P_T y_1 = P_T y_2$. A causal relation R has finite $\mathbf{L}_2$-gain if and only if instead of (3),

$$\|y\|_2 \leq \gamma \|u\|_2 \tag{4}$$

for all $\mathbf{L}_2$ pairs $(u,y) \in R$. The system $\Delta : \mathbf{L}_{2e}^m \to \mathbf{L}_{2e}^m$ (i.e., $p = m$) is said to be *passive* (Vidyasagar, 1981; Willems, 1972) if

$$\int_0^T u(t)^T (\Delta(u))(t)\,dt \geq 0, \tag{5}$$

for all $u \in \mathbf{L}_{2e}$, $T > 0$. Furthermore, it is called *strictly passive* if there exist $\delta > 0$, $\epsilon > 0$ such that

$$\int_0^T u(t)^T (\Delta(u))(t)\,dt \geq \delta \|P_T u\|_2^2 + \epsilon \|P_T \Delta(u)\|_2^2$$

for all $u \in \mathbf{L}_{2e}$, $T > 0$, and *output strictly passive* if this holds with $\delta = 0$. In case Δ is bounded and causal, then passivity is equivalent (van der Schaft, 2017, Proposition 2.2.5) to

$$\int_0^\infty u(t)^T (\Delta(u))(t)\,dt \geq 0 \tag{6}$$

for all $u \in \mathbf{L}_2^m$. (Note that the integral is well-defined because of boundedness of Δ and the Cauchy–Schwarz inequality.) Similarly, in this case Δ is *strictly passive* if there exist $\delta > 0$, $\epsilon > 0$ such that

$$\int_0^\infty u(t)^T (\Delta(u))(t)\,dt \geq \delta \|u\|_2^2 + \epsilon \|\Delta(u)\|_2^2 \quad \forall u \in \mathbf{L}_2^m, \tag{7}$$

and *output strictly passive* if this holds with $\delta = 0$. For later use we also recall the basic property that any output strictly passive system has finite $\mathbf{L}_2$-gain; cf. van der Schaft (2017, Theorem 2.2.13). Like in the $\mathbf{L}_2$-case these passivity notions are directly extended to *relations* $R \subset \mathbf{L}_{2e}^m \times \mathbf{L}_{2e}^m$ satisfying $(0,0) \in R$. Indeed, R is called *strictly passive* if there exist $\delta > 0$, $\epsilon > 0$ such that for all[2] $(u,y) \in R$, $T > 0$

$$\int_0^T u(t)^T y(t)\,dt \geq \delta \|P_T u\|_2^2 + \epsilon \|P_T y\|_2^2, \tag{8}$$

and *output strictly passive* if this holds with $\delta = 0$. Furthermore, a bounded causal relation R is strictly passive if there exist $\delta > 0$, $\epsilon > 0$ such that for all $(u,y) \in R$

$$\int_0^\infty u(t)^T y(t)\,dt \geq \delta \|u\|_2^2 + \epsilon \|y\|_2^2, \tag{9}$$

and output strictly passive if this holds with $\delta = 0$.

The main object of study in this paper is the *feedback interconnection* of two systems $\Sigma_1 : \mathbf{L}_{2e}^{m_1} \to \mathbf{L}_{2e}^{p_1}$ and $\Sigma_2 : \mathbf{L}_{2e}^{m_2} \to \mathbf{L}_{2e}^{p_2}$, with $m_1 = p_2$, $m_2 = p_1$, described by (see Fig. 1)

$$\begin{aligned} u_1 &= e_1 - y_2, & u_2 &= e_2 + y_1, \\ y_1 &= \Sigma_1(u_1), & y_2 &= \Sigma_2(u_2). \end{aligned} \tag{10}$$

[2] Throughout it is assumed that all integrals are well-defined.