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Homogenization of dynamic behaviour of heterogeneous beams with random Young's modulus

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ABSTRACT

The paper deals with the differences between effective and homogeneous solution in case of dynamic of continuous media with random micro-structure. In particular, these differences, called "residuals", are considered for the dynamic linear problem of the Euler-Bernoulli's beam with random Young's modulus.

The differential operator with random coefficient, that describes the eigenvalues problem, is taken into account. The convergence to the effective solution is analysed by introducing two measures: the normalized error between apparent and effective Young's moduli and between the modes shapes.

The obtained results permit to highlight the dependence of the residuals from the micro-structure dimensionless length and the effect of the modes order; these aspects should be considered in the homogenization of dynamic behaviour of random heterogeneous composites.

The assessment of the Rapresentative Volume Element (RVE) by convergence of the Statistical Volume Element (SVE) is also discussed.

1. Introduction

When dealing with a heterogeneous material with periodic micro-structure it is sufficient to analyse the Periodic Unit Cell (PUC) to evaluate the characteristics of the equivalent homogeneous material. In case of random micro-structures it is necessary to individuate the Representative Volume Element (RVE) (Sab, 1992).

Several definitions of RVE have been proposed by researchers for different purposes. Following Hill (1963), RVE is a sample that is representative of the whole mixture in average and contains a sufficient number of inclusions so that its apparent overall moduli are macroscopically uniform. In fact, the RVE is the smallest material volume, sufficiently larger than micro-structure size, for which the spatially overall moduli description is accurate to represent the mean response (Drugan and Willis, 1996).

Introducing the micro-meso, and macroscales principle (Hashin, 1983), the RVE should contain a very large (mathematically infinite) set of microscale elements (e.g. grains) possessing statically homogeneous and ergodic proprieties (Hostogia-Starzewki, 2008). Considering a mesoscale sample with finite dimensions (this is sometimes indicated as Statistical Volume Element - SVE) its propriety are described by the adjective apparent (Huet, 1990) as opposed to effective ones which denote the RVE, so that the problem is to find the scatter and the convergence rate of SVE to RVE.

The homogenization adequacy analysis and the RVE size estimation assumes a great importance from the technical point of view with particular reference to composite materials. The RVE size estimation has been taken into account in several papers (Gitman et al., 2006) (Gitman et al., 2007) (Kanit et al., 2003) also considering the effects on material properties evaluation (He, 2001). This aspect plays an important role in case, for example, of quasi-periodic material homogenization, as masonry with random texture; in this case the "text windows" approach (Cluni and Gusella, 2004) (Gusella and Cluni, 2006) can be used; an alternative approach is the periodization of random media by statically equivalent periodic unit cell (Sab and Nedjar, 2005) (Šejnoha et al., 2008) (Cavalagli et al., 2013).

Moreover it is well known that the analysis of non-local interactions between heterogeneities is necessary when the spatial variation of the considered physical quantities, relative to the micro-structure, cannot be ignored (Luciano and Willis, 2005).

Facing this theme from a mathematical point of view, the homogenization can be regarded as the study of the asymptotic behaviour and convergence of differential operators with variable coefficients. Starting from the initial contributions (Stampacchia, 1965), the literature devoted to homogenization with oscillatory periodic coefficients is

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very wide (Bensoussan et al., 1978) (Bakhvalov and Panasenko, 1989).

Regarding the operators with random coefficients, after the first important solutions obtained by Kozlov (1980) and Papanicolau and Varadhan (1979) several contribution have been reported in literature (Jikov et al., 1994).

Nevertheless the measure of the difference between the heterogeneous solution (apparent solution) and the homogeneous solution (effective solution), that is indicated as *residual* or *corrector*, has had a more limited attention.

At first, the paper of Pozhidaev and Yurinskiĭ (1990) gave some successful attempts in case of second-order elliptic random operators.

The estimation of the discrepancy between solutions has been considered by Bourgeat and Paintniski for a one-dimensional second-order elliptic operators with random coefficients satisfying strong (Bourgeat and Piatnitski, 1999) and different mixing condition (Bourgeat and Piatnitski, 2004). This problem has been examined in depth for long-range correlations by Bal et al. (2008).

In order to analyse the residuals in case of bi-phase medium with different quasi periodic texture, the problem of the convergence rate has been addressed considering the beam model with random Young's modulus (Cluni and Gusella, 2014). Moreover, the analysis has recently been extended to a two-dimensional problem (Cluni and Gusella, 2018).

The homogenization and the RVE estimation assume a relevant importance in dynamic of heterogeneous material (Sanchez-Palencia, 1980) (Bakhvalov and Panasenko, 1989); in particular various research have been dedicated to wave propagation in composite media in low and high frequency range (Parnell and Abrahams, 2006) (Craster et al., 2010).

However, as far as the authors' knowledge is concerned, no results about homogenization residuals for structural dynamic problems have been reported in literature. Vice-versa, this is an important topic related to assess the adequacy of the homogenization analysis. In the present paper, this aspect is dealt with by considering the linear elastic dynamic equation of the Euler-Bernoulli's Beam with random Young's moduls; moreover, the convergence of SVE to RVE, as its dimensions incrase, is considered.

2. Statement of the problem

Observing that "random differential operators", namely differential operators with random coefficients, will be later considered, the following assumptions and definitions have to be introduced.

Let $(\Omega, \Im, P)$ be a standard probability space, i.e. a set Ω equipped with a σ-algebra $\Im$ of measurable subsets and a countably additive non-negative measure P normalized by $P(\Omega) = 1$. We always assume the measure P to be complete (Pankov, 1997).

Let assume that a d -dimensional dynamical system T_x, $x \in \mathbb{R}^d$ is given on Ω (Bourgeat and Piatnitski, 2004), i.e. a family of invertible measurable maps T_x: $\Omega \rightarrow \Omega$, such that.

- $T_{x+y} = T_x T_y$, $T_0 = Id$,
- $P\{(T_x)^{-1}(A)\} = P\{(A)\}$ for any $A \in \Im$ and any $x \in R^d$, that is T_x preserve the measure P,
- T_x is a measurable mapping from $\mathbb{R}^d$ x Ω to Ω where $\mathbb{R}^d$ x Ω is equipped with the product σ-algebra B x $\Im$ and B is the Borel σ-algebra $\mathbb{R}^d$.

For an arbitrary random variable $f = f(\omega)$ we define $f(\omega, z) = f(T_z\omega)$ that is a statistically homogeneous random field; moreover we assume that T_z is an ergodic dynamical system (Bourgeat and Piatnitski, 2004).

Let us consider the flexural vibration of the Euler-Bernoulli's beam, Fig. 1(A). The motion is described by the partial differential equation

$$\frac{\partial^2}{\partial x^2}\left(E(\omega, y)J\frac{\partial^2 v^\varepsilon(x, t)}{\partial x^2}\right) + \rho\frac{\partial^2 v^\varepsilon(x, t)}{\partial t^2} = f(x, t) \tag{1}$$

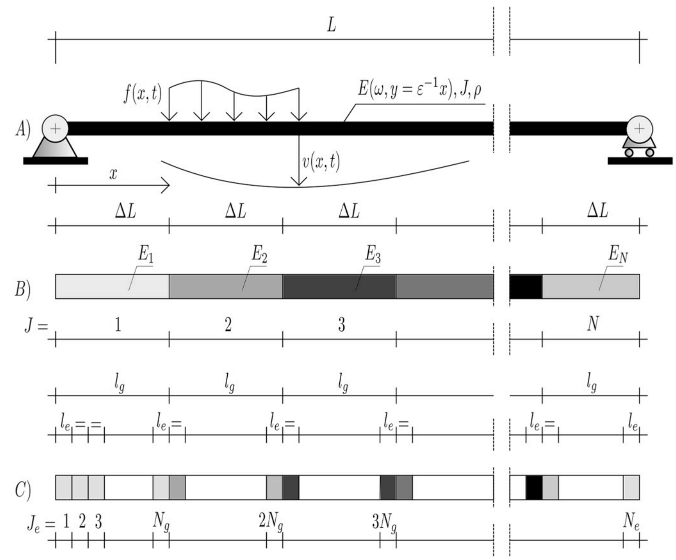

Fig. 1. Mechanical model: (A) Euler-Bernoulli's continuous beam; (B) discrete approach: beam with collinear elements; (C) finite element model.

where $v^\varepsilon(x, t)$ is the stochastic field of the transverse displacement, $0 \le x \le L$, ε is the length parameter that characterizes the microstructure, $y = \varepsilon^{-1}x$ and $E(\omega, y) = E(T_y\omega) = E(T_{x/\varepsilon}\omega)$ Young's modulus is an ergodic homogeneous random field with $0 < c_1 < E(\omega, y) < c_2 < \infty$.

In the previous equation, the ρ linear mass density and the J transversal section inertia moment assume deterministic values constant along x; this assumption is adequate for the composites homogenization where the difference between components stiffness is larger than one between densities. Moreover $f(x, t)$ is the external transverse force that is assumed deterministic. To complete the formulation of the dynamic boundary-value problem, it is necessary to specify the boundary condition and the initial conditions.

The previous equation can be resolved by the superposition of the normal modes.

Regarding the eigenvalue problem, we consider the free vibrations characterised by $f(x, t) = 0$ and the solution becomes separable in space and time

$$v^\varepsilon(x, t) = u^\varepsilon(x)g^\varepsilon(t) \tag{2}$$

whence

$$g^\varepsilon(t)\frac{d}{dx^2}\left(E(\omega, y)\frac{J}{\rho}\frac{d^2u^\varepsilon(x)}{dx^2}\right) + u^\varepsilon(x)\frac{d^2g^\varepsilon(t)}{dt^2} = 0 \tag{3}$$

and

$$-\frac{\frac{d^2g^\varepsilon(t)}{dt^2}}{g^\varepsilon(t)} = \frac{\frac{d}{dx^2}\left(E(\omega, y)\frac{J}{\rho}\frac{d^2u^\varepsilon(x)}{dx^2}\right)}{u^\varepsilon(x)} = \text{cost} = (\lambda^\varepsilon)^2 \tag{4}$$

We obtain the following equations:

$$\frac{d^2}{dx^2}\left(E(\omega, y)\frac{J}{\rho}\frac{d^2u^\varepsilon(x)}{dx^2}\right) - (\lambda^\varepsilon)^2u^\varepsilon(x) = 0 \tag{5}$$

$$\frac{d^2}{dt^2}g^\varepsilon(t) + (\lambda^\varepsilon)^2g^\varepsilon(t) = 0 \tag{6}$$

which correspond to an eigenproblem of a beam with random Young's modulus. Given the boundary conditions, for each choice of $E(\omega, y)$ we obtain the eigenvalues $(\lambda^\varepsilon)_k^2$ (and relative own frequencies $(\lambda^\varepsilon)_k$) and the eigenvectors (normal modes) $u_k^\varepsilon(x)$ with $k = 1,2,3, ...$

The main results on the asymptotic behaviour of random differential operators were first obtained by Kozlov (1980) and Papanicolau and