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On the postulation of lines and a fat line

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ABSTRACT

In the present note we show that the union of r general lines and one fat line in $\mathbb{P}^3$ imposes independent conditions on forms of sufficiently high degree d, where the bound is independent of the number of lines. This extends former results of Hartshorne and Hirschowitz on unions of general lines, and of Aladpoosh on unions of general lines and one double line.

1. Introduction

Let $X \subset \mathbb{P}^n$ be a closed subscheme defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. The Hilbert function of X encodes a number of properties of X and has been classically an object of vivid research in algebraic geometry and commutative algebra. We first recall the definition.

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Definition 1.1 *(Hilbert function).* The Hilbert function of a scheme $X \subset \mathbb{P}^n(\mathbb{K})$ is

$$\mathrm{HF}_X : \mathbb{Z} \ni d \to \dim_{\mathbb{K}}[S(X)]_d \in \mathbb{Z},$$

where $S(X)$ denotes the graded homogeneous coordinate ring of X.

It is well known that the Hilbert function becomes eventually (i.e., for large d) a polynomial. We denote the Hilbert polynomial of X by HP_X. Whereas the Hilbert polynomial can be (in principle) computed algorithmically, the Hilbert function is more difficult to compute. For some varieties, like $\mathbb{P}^n$, the Hilbert function is equal to the Hilbert polynomial, but this behaviour is rare. The next simplest behaviour occurs for subschemes with bipolynomial Hilbert function.

Definition 1.2 *(Bipolynomial Hilbert function).* Following Carlini et al. (2010) we say that X has a *bipolynomial Hilbert function* if

$$\mathrm{HF}_X(d) = \min\{\mathrm{HP}_{\mathbb{P}^n}(d), \mathrm{HP}_X(d)\} \tag{1}$$

for all $d \geqslant 1$.

In other words, X has a bipolynomial Hilbert function if $X \subset \mathbb{P}^n$ imposes the *expected* number of conditions on forms of arbitrary degree $d \geqslant 1$. Essentially by definition, a scheme consisting of general points in $\mathbb{P}^n$ has bipolynomial Hilbert function. An analogous result for X consisting of r general lines in $\mathbb{P}^n$ with $n \geqslant 3$ has been proved by Hartshorne and Hirschowitz (1982, Theorem 0.1). A new proof has been announced recently by Aladpoosh and Catalisano (2017). Recently Carlini et al. (2016) showed that if X consists of r general lines and one general fat point, then, up to a short list of exceptions in $\mathbb{P}^3$, X has bipolynomial Hilbert function, see also Aladpoosh and Ballico (2014) and Ballico (2016).

Aladpoosh (2016) has proved recently that also schemes consisting of general lines and one double line have bipolynomial Hilbert function, with the exception of one double line and two simple lines in $\mathbb{P}^4$ imposing dependent conditions on forms of degree 2. She also conjectured (Aladpoosh, 2016, Conjecture 1.2) that the same holds true for r general lines and one fat flat of arbitrary dimension. In the present note we provide evidence supporting this conjecture for a fat line of arbitrary multiplicity m. Our main result is the following.

Main Theorem. *Let $m \geqslant 1$ be a fixed integer. Then for $d \geqslant d_0(m) := 3\binom{m+1}{3}$, the Hilbert function of a subscheme $X \subset \mathbb{P}^3$ consisting of $r \geqslant 0$ general lines and one line of multiplicity m (i.e., defined by the m-th power of the ideal of a line) satisfies formula (1).*

In other words, a general fat line and an arbitrary number of general lines with multiplicity 1 impose independent conditions on forms of degree $d \geqslant d_0(m)$ (see Theorem 4.1).

It follows from the Serre Vanishing Theorem (Lazarsfeld, 2004, Theorem 1.2.6) that for any subscheme $X \subset \mathbb{P}^n$, there exists a bound $d_0(X)$ such that X imposes independent conditions on forms of degree $d \geqslant d_0(X)$. The point here is that we obtain an explicit bound that depends only on the multiplicity of the fat line and is independent of the number of reduced lines.

We will set up the proof in a way which employs the general strategy of Hartshorne and Hirschowitz (1982) and Carlini et al. (2016). This amounts to work inductively by constructing a suitable sequence of generic subschemes $Z_0, Z_1, \ldots$, along with suitable specializations Z_i' of Z_i. The starting scheme Z_0 consists of the lines in the theorem plus a number of generic points. The essential difficulty in this strategy lies in the choice of the intermediate schemes Z_i and their specializations Z_i'. In our approach this is achieved by using intermediate schemes that contain, apart from disjoint lines and points, also crosses and so-called zig-zags (see Definition 2.4).

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