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The Bordiga surface as critical locus for 3-view reconstructions ☆

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ABSTRACT

In Computer Vision, images of dynamic or segmented scenes are modeled as linear projections from $\mathbb{P}^k$ to $\mathbb{P}^2$. The reconstruction problem consists in recovering the position of the projected objects and the projections themselves from their images, after identifying many enough correspondences between the images. A critical locus for the reconstruction problem is a variety in $\mathbb{P}^k$ containing the objects for which the reconstruction fails. In this paper, we deal with projections both of points from $\mathbb{P}^4$ to $\mathbb{P}^2$ and of lines from $\mathbb{P}^3$ to $\mathbb{P}^2$. In both cases, we consider 3 projections, minimal number for a uniquely determined reconstruction. In the case of projections of points, we declinate the Grassmann tensors introduced in Hartley and Schaffalitzky (2004) in our context, and we use them to compute the equations of the critical locus. Then, given the ideal that defines this locus, we prove that, in the general case, it defines a Bordiga surface, or a scheme in the same irreducible component of the associated Hilbert scheme. Furthermore, we prove that every Bordiga surface is actually the critical locus for the reconstruction for suitable projections. In the case of projections of lines, we compute the defining ideal of the critical locus, that is the union of 3 α-planes and a line congruence of bi-degree $(3, 6)$ and sectional genus 5 in the Grassmannian $\mathbb{G}(1, 3) \subset \mathbb{P}^5$. This last surface is biregular to a Bordiga surface (Verra, 1988). We use this fact to link the two reconstruction problems by showing how to compute the projections of one of the two settings, given the projections of

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the other one. The link is effective, in the sense that we describe an algorithm to compute the projection matrices.

1. Introduction

Linear projections from $\mathbb{P}^3$ to $\mathbb{P}^2$ are the natural geometric model for pictures of static three-dimensional scenes taken from pinhole cameras. Similarly linear projections from $\mathbb{P}^k$ to $\mathbb{P}^2$, with $k \geq 3$, arise when images of particular dynamic and segmented scenes are considered (Wolf and Shashua, 2002; Hartley and Schaffalitzky, 2004; Huang et al., 2002; Fan and Vidal, 2007; Hartley and Vidal, 2004).

Given multiple images of an unknown scene, taken from unknown cameras, the *reconstruction* of the positions of cameras and scene points is a classical problem in Computer Vision, which has been generalized as well in the setting of higher dimensional projective spaces.

Sufficiently many images and sufficiently many sets of corresponding points in the given images should in principle allow for a successful projective reconstruction. Anyway, there exist sets of points, in the ambient space $\mathbb{P}^k$, for which the projective reconstruction fails. These configurations of points are called *critical*, which means that there exist other non projectively equivalent sets of points and cameras that give the same images in the view planes.

Critical loci for projections from $\mathbb{P}^3$ to $\mathbb{P}^2$ have been studied by many authors. Among the many papers on the subject, we recall Buchanan (1988), Krames (1940), Maybank (1992), Hartley (2000), Kahl et al. (2001), Hartley and Kahl (2007), Shashua and Maybank (1996), Åström and Kahl (2003). In the case of projections from higher dimensional $\mathbb{P}^k$ to the projective plane $\mathbb{P}^2$, when $k \geq 4$, critical loci were described in Bertolini and Turrini (2007) in the case of one view, and in Bertolini et al. (2007a, 2007b, 2008, 2009, 2015) in the case of multiple views.

In this paper we focus on the case of three projections from $\mathbb{P}^4$ to $\mathbb{P}^2$, that is the first non-classical case, since three views is the minimum number which allows us to reconstruct the scene, when the scene points are general (in a sense which will be clear later). Our purpose is to get a scheme-theoretical description of the critical locus. The critical locus comes out to be a classical surface in $\mathbb{P}^4$, the so-called Bordiga surface (Bordiga, 1887). The approach used here to obtain the polynomials that generate the ideal of the critical locus is different from the one followed in Bertolini et al. (2015): we use the Grassmann tensor introduced in Hartley and Schaffalitzky (2004). In Åström and Kahl (2003), a first seminal case of this idea has been applied to two projections from $\mathbb{P}^2$ to $\mathbb{P}^1$, while in Bertolini and Magri (2017) this approach is used to study the critical locus when it is a hypersurface. The construction of the critical locus given in Bertolini et al. (2015) allowed the set-theoretical description of it, while the one given here through the Grassmann tensor allows us to compute the generators of the ideal of the critical locus and its first syzygy module, so giving a scheme-theoretical description of the critical locus itself. In more details, the ideal is minimally generated by 4 degree 3 forms that are the maximal minors of a 4×3 matrix with linear entries. From the Hilbert–Burch Theorem, it follows that the critical locus is a determinantal variety of codimension 2 and degree 6 in $\mathbb{P}^4$, and so it belongs to the irreducible component of the Hilbert scheme containing Bordiga surfaces.

A very natural question arising as a consequence of the above results is whether every Bordiga surface in $\mathbb{P}^4$ is the critical locus of suitable projections. To give a positive answer to this question, we heavily use the geometry of Bordiga surfaces.

We recall that a Bordiga surface S is the blow-up of $\mathbb{P}^2$ at 10 general points, embedded in $\mathbb{P}^4$ via the complete linear system of the quartics through the 10 points. S contains exactly 10 lines, corresponding to the base points of the linear system. The ideal of the 10 points is determinantal, too. The 5×4 matrix of linear forms whose maximal minors are the generators of the ideal of the 10 points in $\mathbb{P}^2$ is strongly related to the matrix that presents the surface S in $\mathbb{P}^4$. We use such a relation to prove that the generators of the first syzygy module of the ideal of S can be chosen in a

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