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Imaginary projections of polynomials

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We introduce the imaginary projection of a multivariate polynomial $f \in \mathbb{C}[\mathbf{z}]$ as the projection of the variety of f onto its imaginary part, $\mathcal{I}(f) = \{\mathrm{Im}(\mathbf{z}) : \mathbf{z} \in \mathcal{V}(f)\}$. Since a polynomial f is stable if and only if $\mathcal{I}(f) \cap \mathbb{R}^n_{>0} = \emptyset$, the notion offers a novel geometric view underlying stability questions of polynomials.
We show that the connected components of the complement of the closure of the imaginary projections are convex, thus opening a central connection to the theory of amoebas and coamoebas. Building upon this, the paper establishes structural properties of the components of the complement, such as lower bounds on their maximal number, proves a complete classification of the imaginary projections of quadratic polynomials and characterizes the limit directions for polynomials of arbitrary degree.

1. Introduction

Recent years have seen a lot of interest in stable polynomials, see, e.g., Borcea and Brändén (2008, 2009), Marcus et al. (2015b), Wagner (2011) and the references therein. A polynomial $f = f(\mathbf{z}) = f(z_1, \ldots, z_n) \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, \ldots, z_n]$ is called *stable* if every root $\mathbf{z}$ satisfies $\mathrm{Im}(z_j) \le 0$ for some j. We call f *real stable* if f has real coefficients and is stable.

As recent prominent applications, Marcus, Spielman, and Srivastava employed stable polynomials in the proof of the Kadison–Singer Conjecture (Marcus et al., 2015b) and in the existence proof of families of bipartite Ramanujan graphs of every degree larger than two (Marcus et al., 2015a). Stable

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polynomials have also been used by Borcea and Brändén to prove Johnson's Conjecture (Borcea and Brändén, 2008) and in Gurvits' simple proof of a generalization of van der Waerden's Conjecture for permanents (Gurvits, 2008). Moreover, there are strong connections to hyperbolic polynomials and their hyperbolicity cones, see Section 2.1.

In this paper, we initiate to study the underlying projections on the imaginary parts from a geometric point of view. Given a polynomial $f \in \mathbb{C}[\mathbf{z}]$, introduce the *imaginary projection* of f as

$$\mathcal{I}(f) = \{\mathrm{Im}(\mathbf{z}) : \mathbf{z} \in \mathcal{V}(f)\} \subseteq \mathbb{R}^n ,$$

where $\mathcal{V}(f)$ denotes the variety of f and $\mathrm{Im}(\mathbf{z}) = (\mathrm{Im}(z_1), \ldots, \mathrm{Im}(z_n))$. So, in particular, f is stable if and only if

$$\mathcal{I}(f) \cap \mathbb{R}^n_{>0} = \emptyset .$$

Our work is motivated by the theory of *amoebas* as well as by the general goal to reveal and understand convexity phenomena in algebraic geometry, see Blekherman et al. (2013). Amoebas are the images of algebraic varieties in the algebraic torus $(\mathbb{C}^*)^n$ under the log-absolute map:

$$\mathcal{A}(f) = \{(\log|z_1|, \ldots, \log|z_n|) : \mathbf{z} \in \mathcal{V}(f) \cap (\mathbb{C}^*)^n\} \subseteq \mathbb{R}^n ,$$

see Gelfand et al. (1994). Coamoebas employ the arg-map rather than the log-absolute map; see, e.g., Forsgård (2015).

For amoebas, important structural results as well as their occurrences in a broad spectrum of mathematical disciplines have been intensively studied, see Mikhalkin (2004), Passare and Rullgård (2004), Passare and Tsikh (2005) as well as the recent survey de Wolff (2017). For coamoebas, investigations are much more recent (Forsgård, 2015; Forsgård and Johansson, 2015; Nisse and Sottile, 2013). A prominent result states that the complement of an amoeba as well as the complement of the closure of a coamoeba consists of finitely many convex components, see Forsgård and Johansson (2015), Gelfand et al. (1994). As a key result, which also motivates our study, we show that the closure of the complement of the imaginary projection of a polynomial consists of finitely many convex components as well, see Theorem 4.1.

While there are important analogies among amoebas, coamoebas, and imaginary projections, there are also fundamental differences between these structures. The fibers of the log-absolute maps underlying amoebas are compact, whereas for imaginary projections they are not compact. Furthermore, the limit directions of amoebas, also known as tentacles, are characterized by the logarithmic limit sets and thus carry a polyhedral structure; see Maclagan and Sturmfels (2015, Theorem 1.4.2). In contrast, the limit directions of the imaginary projections are not polyhedral in general, see Section 6. For coamoebas, which are defined on a torus, Nisse and Sottile have introduced a variant of the logarithmic limit sets, by considering accumulation points of arguments of sequences with unbounded logarithm (Nisse and Sottile, 2013).

Building upon the fundamental convexity result, we study structural properties of imaginary projections. We also give lower bounds on the maximal number of components of the complement, see Corollary 4.5.

We investigate important subclasses, such as quadratic and multilinear polynomials. For the class of real quadratic polynomials, we can provide a complete classification of the imaginary projections, see Theorem 5.4. Indeed, this classification result in Theorem 5.4 is somewhat unexpected, since it involves various qualitatively different cases.

Starting from the well-known results on tentacles of amoebas, we characterize the limit points of the imaginary projections. Contrary to the case of the amoeba of a non-zero polynomial f, it is possible that every point on the sphere $\mathbb{S}^{n-1}$ is a limit direction of the imaginary projection of f. For $f \in \mathbb{C}[\mathbf{z}]$, we provide a criterion for one-dimensional families of limit directions at infinity. In the case $n = 2$ this also characterizes the situations that all points are limit points. See Theorem 6.5 and Corollary 6.7 for further details.

It is easy to see that real projections of complex polynomials should behave in the same way as imaginary projections, since one projection is easily seen to be an instance of the other by replacing