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On generating sets of the clone of aggregation functions on finite lattices

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ABSTRACT

In a recent paper [12] we have shown that aggregation functions on a bounded lattice L form a clone, i.e., the set of functions closed under projections and composition of functions. Moreover, for any finite lattice L we gave a finite set of unary and binary aggregation functions on L from which the aggregation clone is generated.

In this paper, a general method for constructing generating sets of the aggregation clone on L is presented. Our approach is based on extending of L-valued capacities leading to so-called full systems of aggregation functions. Several full systems on L are presented (including singleton ones) and their arities are discussed.

1. Introduction

Clones acting on non-empty sets belong to basic algebraic structures [5,15] and they capture a lot of properties important in various applications, especially in discrete mathematics and information sciences. For example, many natural problems studied in discrete mathematics and artificial intelligence can be modeled as so-called constraint satisfaction problems (CSP, for short), see e.g., [6]. Although these are usually of a hard complexity, nevertheless at least some of them have been successfully solved by applying a clone theory.

In particular, in recent years, clones on bounded chains and on bounded lattices considering monotone functions satisfying boundary conditions, i.e., considering aggregation functions (of an arbitrary arity), are of interest, cf [2,8,17]. So, for example, when considering as the underlying career $L = [a, b]$ a non-trivial interval of reals, then all continuous aggregation functions invariant under transform by means of an arbitrary automorphisms $\phi\colon L \to L$, were characterized by Ovchinnikov and Dukchovny [16] as a polynomial clone of L (for more details on polynomial clones, see Section 2). Note that this clone is isomorphic to the clone of monotone idempotent Boolean functions. Some years later, all continuous aggregation functions on $[a, b]$ invariant under any monotone bijection $\tau\colon [a, b] \to [a, b]$ were characterized by Bronevich and Mesiar [4], to be isomorphic to the clone of self-dual monotone idempotent Boolean functions (observe that this clone is generated by the ternary median, see [13]). For some latest studies concerning clones of aggregation functions see e.g., [3,10–12].

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In [12] we have shown that for lattices with at least three elements, any set of unary aggregation functions together with the lattice operations is not enough to generate the full aggregation clone. Consequently, at least one aggregation function of arity at least two (and different from the lattice operations) must be present in any generating set.

The aim of this contribution is a deeper study of particular clones of aggregation functions on finite lattices. We present a general method for constructing generating sets of the aggregation clone. Our approach is based on extending of L-valued capacities, cf [14]., leading to the so-called full systems of aggregation functions. We show that aggregation functions from any full system together with certain unary aggregation functions and the lattice operations already form a generating set of the aggregation clone.

In comparison with the result presented in [12], where only one particular type of generating set of the aggregation clone was presented, the method based on full systems allows to find new families of generating sets. This more flexible method also provides a better upper bound for the number of generators of arity at most two than that which has been found in [12]. Moreover, a straightforward modification of the binary case to higher arities is discussed as well.

The paper is organized as follows. More details concerning algebraic preliminaries are given in the next section, bringing also several interesting examples. Section 3 contains our main results concerning the generating sets of the aggregation clone, focusing on aggregation functions with arity $n = 2$. In Section 4, higher arities of aggregation functions in generating sets are considered. Finally, some concluding remarks are added.

2. Algebraic preliminaries

Recall that a *lattice* L is a partially ordered set (a poset, in brief) $(L, \leq)$, where for every pair of elements $a, b \in L$ there exists their supremum $a \vee b$ and infimum $a \wedge b$ (with respect to the partial order $\leq$). Equivalently, any lattice can be viewed as an algebra $(L; \vee, \wedge)$ with two binary operations $\vee$ and $\wedge$ representing suprema and infima. A lattice L is *bounded* whenever it has the least element 0 and the greatest element 1, and *distributive* if it fulfills the distributive identity $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (or, equivalently, its dual identity), see e.g., [9].

By a *sublattice* of L is meant any subset $M \subseteq L$ which is closed under suprema and infima, i.e., $a \vee b, a \wedge b \in M$ for all $a, b \in M$. As the intersection of sublattices is again a sublattice, for any subset X of L there is a least sublattice containing X, the so-called sublattice generated by X. Recall that for any non-void subset X of L, the sublattice of L generated by X consists of elements of the form $q(x_1, \ldots, x_n)$ for any lattice polynomial q and $x_1, \ldots, x_n \in X$.

A *clone* on a non-void set A is a set of (finitary) operations on A which contains all the projection operations on A and that is closed with respect to the composition.

Projections and composition of functions are formally defined as follows: let $A \neq \emptyset$ be a set and $n \in \mathbb{N}$ be a positive integer. For any $i \leq n$, the ith n-ary projection is for all $(x_1, \ldots, x_n) \in A^n$ defined by

$$p_i^n(x_1, \ldots, x_n) := x_i.$$

Composition forms from one k-ary operation f: $A^k \to A$ and k n-ary operations $g_1, \ldots, g_k : A^n \to A$, an n-ary operation $f(g_1, \ldots, g_k) : A^n \to A$ defined by

$$f\big(g_1, \ldots, g_k\big)(x_1, \ldots, x_n) := f\big(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)\big), \tag{1}$$

for all $(x_1, \ldots, x_n) \in A^n$.

Obviously, for a given set $A \neq \emptyset$, the set consisting of all the projections (of all finite arities) forms a clone on A, which is contained in any other clone. Thus this is the least clone with respect to set inclusion. The greatest one, called the *full clone* on A, contains all the operations on A. Further, it can be easily seen that the system of all clones on A forms an intersection-closed family, i.e., for any indexed system C_i, $i \in I$ (I a nonempty index set) of clones on A, their intersection $\bigcap_{i \in I} C_i$ is again a clone. Hence for any set F of operations defined on A there exist the least clone $[F]$ containing the set F. Let us recall that $[F] = \bigcap\{C : C \text{ clone on } A,\ F \subseteq C\}$. If C is a clone and $C = [F]$ for some set of operations F, we say that C *is generated* by F. Consequently, the system of all clones on A forms a complete lattice with respect to set inclusion, where the infimum operation is the intersection, while the supremum of any family of clones is equal to the clone generated by the union of the respective family.

Recall that an *aggregation function* on a bounded lattice L is a function A: $L^n \to L$ that

(i) is nondecreasing (in each variable), i.e., for any $\mathbf{x}, \mathbf{y} \in L^n$:

$$A(\mathbf{x}) \leq A(\mathbf{y}) \quad \text{whenever} \quad \mathbf{x} \leq \mathbf{y},$$

(ii) fulfills the boundary conditions

$$A(0, \ldots, 0) = 0 \quad \text{and} \quad A(1, \ldots, 1) = 1.$$

The integer n represents the arity of the aggregation function.

It is not hard to show that the following sets of functions form a clone on any bounded lattice L:

- $\mathsf{Bound}(L)$, the set of all *bounds-preserving functions* on L, i.e., $f(0, \ldots, 0) = 0$ and $f(1, \ldots, 1) = 1$,
- $\mathsf{Mon}(L)$, the set of all *nondecreasing functions* on L, i.e., $f(\mathbf{x}) \leq f(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in L^n$ with $\mathbf{x} \leq \mathbf{y}$,
- $\mathsf{Pol}(L)$, the set of all *polynomial functions* on L, i.e., the clone generated by lattice operations $\wedge$, $\vee$,