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Binary simple homogeneous structures

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We describe all binary simple homogeneous structures $\mathcal{M}$ in terms of $\emptyset$-definable equivalence relations on M, which "coordinatize" $\mathcal{M}$ and control dividing, and extension properties that respect these equivalence relations.

1. Introduction

We describe the fine structure of binary simple homogeneous structures to the extent that seems feasible without further assumptions and with known concepts and methods from infinite model theory. In this respect, this article completes the earlier work on this topic by Aranda Lopéz [3], Ahlman [2] and the present author [2,19–21]. Before discussing the results, we explain what "homogeneity" means here, and give some background.

We call a structure $\mathcal{M}$ *homogeneous* if it is countable, has a finite relational vocabulary (also called signature) and every isomorphism between finite substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$. For a countable structure $\mathcal{M}$ with finite relational vocabulary, being homogeneous is equivalent to having elimination of quantifiers [16, Corollary 7.42]; it is also equivalent to being a *Fraïssé limit* of an *amalgamation class* of finite structures [10,16]. A structure with a relational vocabulary will be called *binary* if every relation symbol is unary or binary. Certain kinds of homogeneous structures have been classified. This holds for homogeneous partial orders, graphs, directed graphs, finite 3-hypergraphs, and coloured multipartite graphs [4,12,13,22,25,27,26,31,32]. For a survey about homogeneous structures, including their connections to permutation groups, Ramsey theory, topological dynamics and constraint satisfaction problems, see [29] by Macpherson.

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A detailed theory, due to Lachlan, Cherlin, Harrington, Knight and Shelah [5,17,22–24], exists for stable infinite homogeneous structures, for any finite relational language, which describes them in terms of (finitely many) dimensions and $\emptyset$-definable indiscernible sets (which may live in $\mathcal{M}^{\mathrm{eq}}$); see [23] for a survey. This theory also sheds light on finite homogeneous structures. But we seem to be a very long way from a classification of (even binary) finite homogeneous structures. This has consequences for (eventual) classifications of infinite homogeneous structures, for the following reason. Suppose that $\mathcal{N}$ is a finite (binary) homogeneous structure. Let $\mathcal{M}$ be the disjoint union of ω copies of $\mathcal{N}$ and add an equivalence relation such that each equivalence class is exactly the set of elements in some copy of $\mathcal{N}$. Then $\mathcal{M}$ is a (binary) stable homogeneous structure. Hence a classification of all (binary) stable homogeneous structures presupposes an equally detailed classification of all (binary) finite homogeneous structures. Thus we ignore the inner structure of such ("very local") finite "blocks" as the copies of $\mathcal{N}$ in the example, and focus on the "global fine structure" of an infinite structure $\mathcal{M}$.

The notion of simplicity generalizes stability and implies that there is a quite useful notion of independence. Moreover, there are interesting (binary) simple homogeneous structures which are unstable, such as the Rado graph and (other) homogeneous metric spaces with a finite distance set. (More about this is Section 7.4.) From this point of view it is natural, and seems feasible, to study simple homogeneous structures. From now on when saying that a structure is simple we assume that it is infinite, so "simple and homogeneous" implies that it is countably infinite. The theory of binary simple homogeneous structures has similarities to the theory of stable homogeneous structures, but also differences. Every stable (infinite) homogeneous structure is ω-stable, hence superstable, with finite SU-rank (which is often called U-rank in the context of stable structures). Analogously, every binary simple homogeneous structure is supersimple with finite SU-rank (which is bounded by the number of 2-types over $\emptyset$) [19]. However, the rank considered in the work on stable homogeneous structures is Shelah's "$CR(\ ,2)$-rank" [33, p. 55]. This rank is finite for stable homogeneous structures, but it is infinite for the Rado graph. If $\mathcal{M}$ is stable (infinite) and homogeneous and $C \subseteq M^{\mathrm{eq}}$ is $\emptyset$-definable and such that, on C, there is no $\emptyset$-definable nontrivial equivalence relation, then C is an indiscernible set. This is not true in general for (binary) simple homogeneous structures, as witnessed again by the Rado graph.

Suppose that $\mathcal{M}$ is binary, simple, and homogeneous. We already mentioned that $Th(\mathcal{M})$, the complete theory of $\mathcal{M}$, is supersimple with finite SU-rank. It is also known that $Th(\mathcal{M})$ is 1-based and has trivial dependence/forking [21, Fact 2.6 and Remark 6.6]. If $\mathcal{M}$ is, in addition, *primitive*, then $\mathcal{M}$ has SU-rank 1 and is a random structure [21]. (See Section 2.3 for a definition of 'primitive structure'.) Before stating the main results of this article, we note that, although the definition (above) of 'homogeneous structure' involves the assumption that the structure is countable, the main results hold for *every* model of $Th(\mathcal{M})$. The reason is that, $\mathcal{M}$ (being homogeneous) is ω-categorical and hence ω-saturated. So if elements could be found in some $\mathcal{N} \models Th(\mathcal{M})$ such that one of the statements (a)–(d) below fails in $\mathcal{N}$, then such elements could also be found in $\mathcal{M}$.

Main results *(Theorems 5.1 and 6.2). Suppose that $\mathcal{M}$ is binary, simple, and homogeneous (hence supersimple with finite SU-rank and trivial dependence). Let* $\mathbf{R}$ *be the (finite) set of all $\emptyset$-definable equivalence relations on M. If $a \in M$ and $R \in \mathbf{R}$, then a_R denotes the R-equivalence class of a as an element of M^{eq}.*

(a) Coordinatization by equivalence relations: *For every $a \in M$, if $\mathrm{SU}(a) = k$, then there are $R_1, \ldots, R_k \in \mathbf{R}$, depending only on $\mathrm{tp}(a)$, such that $a \in \mathrm{acl}(a_{R_k})$, $\mathrm{SU}(a_{R_1}) = 1$, $R_{i+1} \subset R_i$ and $\mathrm{SU}(a_{R_{i+1}}/a_{R_i}) = 1$ for all $1 \le i < k$ (or equivalently, $\mathrm{SU}(a/a_{R_i}) = k - i$ for all $1 \le i \le k$).*

(b) Characterization of dividing: *Suppose that $a, b, \bar{c} \in M$ and $a \not\mathop{\smile}\limits^{\vert}_{\bar{c}} b$. Then there is $R \in \mathbf{R}$ such that $a \not\mathop{\smile}\limits^{\vert}_{\bar{c}} a_R$ and $a_R \in \mathrm{acl}(b)$ (and thus $a_R \notin \mathrm{acl}(\bar{c})$).*