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What we talk about when we talk about numbers ☆

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ABSTRACT

In this paper, I describe and motivate a new species of mathematical structuralism, which I call *Instrumental Nominalism about Set-Theoretic Structuralism*. As the name suggests, this approach takes standard Set-Theoretic Structuralism of the sort championed by Bourbaki, and removes its ontological commitments by taking an instrumental nominalist approach to that ontology of the sort described by Joseph Melia and Gideon Rosen. I argue that this avoids all of the problems that plague other versions of structuralism.

0. Introduction

In this paper, I'd like to describe and motivate a new species of mathematical structuralism. In the philosophy of mathematics, structuralism is a genus of theses concerning the subject matter and ontology of mathematics, as well as the correct semantics for mathematical language. Each species that belongs to that genus is motivated by the observation that mathematicians are agnostic about the intrinsic or internal nature of the objects that they study. In this sense, structuralism is very much a philosophy of mathematics that is inspired by and guided by mathematical practice. Mathematicians are indifferent to the non-mathematical features of the objects they study. They care only about the so-called structural features of those objects. For instance, they care that 2 is less than 3 and that π is transcendental. They do not care whether 2, 3, or π is a set or a class of sets, a Dedekind cut in the rationals or an equivalence class of

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Cauchy sequences of rationals, a universal or a particular, an abstract object or a concrete one, a necessary existent or an entity that exists only contingently, and so on. But, while each species of structuralism agrees on this indifference, they differ significantly on the ontology and semantics of mathematics that best accommodates it.

Why offer a new species of structuralism when the genus is already so crowded? The subject matter of mathematics, together with the semantics of mathematical language, has an extensive job description. There are many boxes that any candidate ontology and semantics would ideally tick. As we will see, while each of the existing species of structuralism ticks many of these boxes, they all leave many untouched. I hope that my new version will tick all of the boxes.

My strategy is as follows: I will begin with a species of structuralism as different from my final proposal as can be. Then I will raise an objection to that species that will lead us to formulate a new species that avoids the objection. But now I will note a objection to this new species, and I will formulate a further new species that avoids both objections. And so on. I will repeat this process until we arrive at our new version of structuralism, which avoids all objections.

Before we start, a disclaimer: this paper covers quite a lot of ground. Each new tickbox in the job description for the subject matter of mathematics deserves, and has received, much more detailed discussion than I am able to give it here. But this paper is programmatic — my purpose is to motivate moving to a new version of structuralism. So I hope readers will forgive me if I reject their favoured version of structuralism without the full discussion they would wish.

1. Structuralism and the axiomatic method

As John P. Burgess [7] argues in detail, structuralism in the philosophy of mathematics is the inevitable response to the introduction of the axiomatic method as the fundamental methodology of mathematics towards the end of the nineteenth century. And the axiomatic method was, in turn, the inevitable conclusion of the quest for greater rigour in mathematics and the attempt to expel geometric, spatial, and other forms of intuition from mathematical proofs and definitions. According to the axiomatic method, each area of mathematics — real or complex analysis, probability or measure theory, group theory, number theory, graph theory, linear algebra, topology, and so on — is characterized by a set of axioms. These pick out the items of interest in that area — the real numbers, the complex field, the probability spaces, the groups, the natural numbers, the graphs, the vector spaces, the topological spaces, and so on. They do this by spelling out the properties shared by all of the items of interest.

If we take a set-theoretic approach, the items of interest are *systems*. In this context, a system consists of an underlying set or a family of underlying sets, perhaps equipped with some distinguished elements of those sets, distinguished functions involving those sets, and relations amongst the members of those sets. Thus, for instance, Cayley's group axioms characterise the subject matter of group theory [11]. They apply to systems $(G, e, *)$, where e is a distinguished element of the underlying set G, and $*$ is a binary function on G. Similarly, Dedekind's axioms for a complete ordered field characterise the subject matter of real analysis [13]. They apply to systems $(R, 0, 1, +, \times, <)$, where 0, 1 are distinguished elements of R, $+$ and $\times$ are binary functions on R, and $<$ is a binary relation on R. And Peano's axioms for a vector space characterise the subject matter of linear algebra [37]. They apply to systems $(V, K, \mathbf{0}, +_V, 0, 1, +_K, \times_K, \cdot)$, where $\mathbf{0}$ is a distinguished element of V, $+_V$ is a binary function on V, 0, 1 are distinguished elements of K, $+_K$, $\times_K$ are binary functions on V, and $\cdot$ is a function defined on $K \times V$. And so on.

On the other hand, if we take a category-theoretic approach, the items of interest are *categories*, or *objects* in *categories*. There are (at least) four ways in which we might formulate the axiomatic method on this approach. On the first, which we might call the *category-based approach*, the axioms characterise a certain sort of category, saying that the items of interest in the area of mathematics in question are all and only the categories of this sort. This is close to the set-theoretic approach, except that the axioms are stated in

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