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On Young's inequality

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A B S T R A C T

We present some inequalities for trigonometric sums. Among others, we prove the following refinements of the classical Young inequality.

(1) Let $m \geq 3$ be an odd integer, then for all $n \geq m-1$,

$$\sum_{k=1}^{n} \frac{\cos(k\theta)}{k} \geq \sum_{k=1}^{m} \frac{(-1)^k}{k}.$$

The sign of equality holds if and only if $n = m$ and $\theta = \pi$. The special case $m = 3$ is due to Brown and Koumandos (1997).

(2) For all even integers $n \geq 2$ and real numbers $r \in (0, 1]$ and $\theta \in [0, \pi]$ we have

$$\sum_{k=1}^{n} \frac{\cos(k\theta)}{k} r^k \geq -\frac{5}{48}(5+\sqrt{5}) = -0.75375....$$

The sign of equality holds if and only if $n = 4$, $r = 1$ and $\theta = 4\pi/5$. We apply this result to prove the absolute monotonicity of a function which is defined in terms of the log-function.

1. Introduction and statement of main results

In 1912, W.H. Young [11] published interesting inequalities for the cosine polynomial

$$C_n(\theta) = \sum_{k=1}^{n} \frac{\cos(k\theta)}{k}.$$

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Among others, he proved that for all integers $n \geq 2$ and real numbers $\theta \in [0, \pi]$ we have

$$C_n(\theta) > -1. \tag{1.1}$$

This inequality is one of the classical results in the theory of trigonometric polynomials. It has attracted the attention of many researchers who presented numerous extensions, variants, related results and applications to geometric function theory and other branches. For more information on this subject we refer to Askey [1], Askey and Gasper [2], Koumandos [7], and Milovanović et al. [8].

We discovered only recently that there is an error in Young's original proof of (1.1). He first showed that C_n takes its minimum precisely at $\theta = \pi$ if n is odd, and at $\theta = \pi - \pi/(n+1)$ if n is even. Then, he noted that

$$C_n(\pi) > -1 \quad \text{for odd } n.$$

For even n, he proved that

$$C_n\Big(\pi - \frac{\pi}{n+1}\Big) > -1$$

using an incorrect claim, namely, that

$$C_n\Big(\pi - \frac{\pi}{n+1}\Big) = \sum_{k=1}^{n/2} (-1)^k u_k v_k \tag{1.2}$$

with

$$u_k = \frac{1}{k} - \frac{1}{n+1-k} \quad \text{and} \quad v_k = \cos\Big(\frac{k\pi}{n+1}\Big). \tag{1.3}$$

From

$$u_1 > u_2 > \cdots > u_{n/2} > 0 \quad \text{and} \quad v_1 > v_2 > \cdots > v_{n/2} > 0, \tag{1.4}$$

he concluded that

$$C_n\Big(\pi - \frac{\pi}{n+1}\Big) \geq -u_1 v_1 = -\Big(1 - \frac{1}{n}\Big) \cos\Big(\frac{\pi}{n+1}\Big) > -1. \tag{1.5}$$

Indeed, formula (1.2) only holds when we replace u_k by

$$u_k^* = \frac{1}{k} + \frac{1}{n+1-k}\,.$$

Although (1.4) remains valid, (1.5) becomes

$$C_n\Big(\pi - \frac{\pi}{n+1}\Big) \geq -u_1^* v_1 = -\Big(1 + \frac{1}{n}\Big) \cos\Big(\frac{\pi}{n+1}\Big) > -1,$$

but the inequality

$$-u_1^* v_1 > -1$$

is only valid for $n = 2$. It is false for $n = 4, 6, 8, \ldots$. This means that Young's proof is still erroneous if we just correct the sign error in (1.3).