ARTICLE IN PRESS YACHA:1188

Appl. Comput. Harmon. Anal. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Limits of calculating the finite Hilbert transform from discrete samples ☆,☆☆

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ARTICLE INFO

Article history:
Received 26 July 2016
Received in revised form 14 March 2017
Accepted 19 March 2017
Available online xxxx
Communicated by Charles K. Chui

MSC:
41A65
44A15
30E20
65R10

Keywords:
Approximation
Hilbert transform
Sampling
Strong divergence

ABSTRACT

This paper studies the problem of calculating the finite Hilbert transform $\widetilde{f} = \mathrm{H}f$ of functions f from the set $\mathcal{B}$ of continuous functions with a continuous conjugate $\widetilde{f}$ based on discrete samples of f. It is shown that all sampling based linear approximations which satisfy three natural axioms diverge strongly on $\mathcal{B}$ in the uniform norm. More precisely, we consider sequences $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of linear approximation operators $\mathrm{H}_N : \mathcal{B} \to \mathcal{B}$ such that the calculation of $\mathrm{H}_N f$ is based on discrete samples of f and which satisfies two additional natural axioms. We show that for all such sequences $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ there always exists an $f \in \mathcal{B}$ such that $\lim_{N\to\infty} \|\mathrm{H}_N f\|_\infty = +\infty$. Moreover, it is shown that on the subset $\mathcal{B}^{1/2}$ of all $f \in \mathcal{B}$ with finite Dirichlet energy an even larger class of sampling based approximation sequences diverges weakly, i.e. for all such sequences there always exists an $f \in \mathcal{B}^{1/2}$ such that $\limsup_{N\to\infty} \|\mathrm{H}_N f\|_\infty = +\infty$.

1. Introduction

The finite Hilbert transform $\mathrm{H} : f \mapsto \widetilde{f}$ relates a function f, defined on the unit circle, and its conjugate $\widetilde{f}$ by the principal value integral

$$\widetilde{f}(\mathrm{e}^{\mathrm{i}\theta}) = (\mathrm{H}f)(\mathrm{e}^{\mathrm{i}\theta}) = \lim_{\epsilon\to 0} \frac{1}{2\pi} \int\limits_{\epsilon\leq|\theta-\tau|\leq\pi} \frac{f(\mathrm{e}^{\mathrm{i}\tau})}{\tan([\theta-\tau]/2)}\,\mathrm{d}\tau \ , \quad \theta \in [-\pi,\pi) \ . \tag{1}$$

☆ This work was partly supported by German Research Foundation (DFG) under grant BO 1734/20-1 and PO 1347/3-1.
☆☆ Part of this work was presented at the *2017 American Control Conference (ACC)*, Seattle, WA, USA, May 2017 [8].
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http://dx.doi.org/10.1016/j.acha.2017.03.002
1063-5203/

This transformation plays an important role in science and engineering since it is closely related to our causal perception of the physical world [13,23]. If, for example, a discrete sequence is known to be causal then the real and imaginary part of its Fourier transform are related by (1). For signals, defined on the real axis, the corresponding dependency is also known as Kramers–Kronig relation. It allows, in principle, to retrieve the phase of a causal signal from amplitude measurements in situations where it is hard or impossible to obtain the phase information directly [16].

Because of its practical importance, there are many different approaches and algorithms to evaluate numerically the Hilbert transform [29,30,18,17,19,24,26,21]. Thereby, the principal value integral (1) constitutes the major problem in evaluating $\mathrm{H}f$ numerically. Consequently, there has been significant research effort to find appropriate quadrature formulas for the integral in (1) using Gaussian, Chebychev, or spline interpolating functions, to mention just a few (see, e.g., the overview in [18, Vol. 1, Chapter 14]). Other approaches [30] are based on expanding $\mathrm{H}f$ in the eigenfunction of H. The corresponding expansion coefficients are given by integrating f, which is again approximated by an appropriate quadrature formula. Pointwise convergence results for such quadrature formulas have been shown for Hölder continuous functions [9,19], whereas uniform convergence of Hilbert transform approximations were reported for analytic function [15,14].

It is an interesting question whether the smoothness assumptions are necessary to obtain these pointwise or uniform convergence results or whether there exist approximations which will converge even for non-smooth continuous functions. This questions is investigates in this paper for a very general class of approximation methods. Note that all above discussed approximation methods resting upon numerical quadrature formulas. Therefore, all of these approaches are based on a finite set of discrete samples of $\{f(\zeta_n)\}_{n=1}^N$ of f. This property is necessary in order to use digital computers for the numerical evaluation. To generalize these approximation approaches, assume that a set of sampling points $\{\zeta_n\}_{n=1}^N$ is given. Then one tries to design a sequence of linear operators $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$, each of which is concentrated on the samples $\{f(\zeta_n)\}_{n=1}^N$, such that $\mathrm{H}_N f$ approximates $\mathrm{H}f$ sufficiently well if N gets sufficiently large. In other words, one tries to find a sequence $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ of bounded linear operators such that

$$\lim_{N\to\infty} \left\|\mathrm{H}_N f - \mathrm{H}f\right\|_\infty = 0 \qquad \text{for all } f \in \mathcal{B}\,,$$

where $\mathcal{B}$ is the Banach space of all continuous functions with a continuous conjugate and where we are interested to approximate $\mathrm{H}f$ in the uniform norm. It was already shown [5] that for every sampling based approximation method $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ one has

$$\limsup_{N\to\infty} \left\|\mathrm{H}_N f - \mathrm{H}f\right\|_\infty = +\infty \qquad \text{for some } f \in \mathcal{B}\,. \tag{2}$$

However, if $\{\mathrm{H}_N\}_{N\in\mathbb{N}}$ satisfies (2) then this implies only the existence of a “bad subsequence” $\{N_k(f)\}_{k\in\mathbb{N}}$ such that $\mathrm{H}_{N_k} f$ diverges as $k\to\infty$. Nevertheless, there may exist “good subsequences” $N_k(f)$ such that $\mathrm{H}_{N_k(f)} f$ converges to $\mathrm{H}f$. The interesting question is then, whether it is always possible to find such a good subsequence $\{N_k(f)\}_{k\in\mathbb{N}}$. This paper will prove that the answer is negative for a very general class of sampling based Hilbert transform approximation operators. Since the convergent subsequence $\{N_k(f)\}_{k\in\mathbb{N}}$ depends generally on the actual function f, this problem can be related to the non-existence of adaptive algorithms to calculate the Hilbert transform [6].

Divergence results of the form (2), which involve a lim sup-divergence, are usually proven by showing that the operator norms $\|\mathrm{H}_N\|$ are not uniformly bounded and by applying the Banch–Steinhaus theorem [1]. However, to prove that there exist no convergent subsequence, one has to verify that

$$\lim_{N\to\infty} \left\|\mathrm{H}_N f - \mathrm{H}f\right\|_\infty = +\infty \qquad \text{for some } f \in \mathcal{B}\,,$$