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A meshless method for solving the nonlinear inverse Cauchy problem of elliptic type equation in a doubly-connected domain

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ABSTRACT

In the paper a nonlinear inverse Cauchy problem of nonlinear elliptic type partial differential equation in an arbitrary doubly-connected plane domain is solved using a novel meshless numerical method. The unknown Dirichlet data on an inner boundary are recovered by over-specifying the Cauchy data on an outer boundary. A homogenization function is derived to annihilate the Cauchy data on the outer boundary, and then a homogenization technique generates a transformed equation in terms of a transformed variable, whose outer Cauchy boundary conditions are homogeneous. When the numerical solution is expanded by a sequence of boundary functions, which automatically satisfy the homogeneous Cauchy boundary conditions on the outer boundary, we can solve the transformed equation by the domain type meshless collocation method. For the nonlinear inverse Cauchy problems we require to iteratively solve the linear systems with the right-hand sides varying per iteration step. The accuracy and robustness of the homogenization boundary function method (HBFM) are examined through seven numerical examples, where we compare the exact Dirichlet data on the inner boundary to the ones recovered by the HBFM under a large noisy disturbance.

1. Introduction

The numerical solution of the oncoming nonlinear inverse Cauchy problem is to solve the boundary value problem (BVP) of a nonlinear elliptic type partial differential equation (PDE) in a doubly-connected domain, given the over-specified Cauchy data on an accessible outer boundary. For the purpose of data completion one needs to recover the unknown data on an inaccessible inner boundary. However, the nonlinear inverse Cauchy problem is a difficult issue, since the solution may not depend continuously on the given data, where the error in the input data leads to an incorrect numerical solution.

The inverse Cauchy problem is not an artificial and fictitious problem without involving the physical meaning. The Cauchy boundary conditions are often encountered in the applications of non-destructive testing to solid materials. In a practical application, the electrostatic image used in the non-destructive testing of metallic plates leads to an inverse Cauchy problem in two-dimension. In order to detect the unknown shape of an inclusion within a conducting metal, the over-specified Cauchy

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data (the voltage and current) are imposed on the accessible outer boundary, which requires to solve an inverse Cauchy problem from the available boundary data measured on the outer boundary.

Because the inverse Cauchy problems of elliptic type PDEs are highly ill-posed in nature [1], they have been solved by adopting different numerical methods [2–23]. The most of them only considered the linear inverse Cauchy problems for the linear PDEs, including the Laplace equation, the Poisson equation, the Helmholtz equation, and the modified Helmholtz equation, etc. In the literature there are only a few papers to solve the nonlinear inverse Cauchy problems [24–27], which are much more difficult than that for the linear inverse Cauchy problems. When the nonlinear inverse Cauchy problem is solved, the nonlinearity may render a highly ill-posed behavior in the numerical method.

The meshless methods, which do not require extensive meshes and elements generation and are flexible to deal with the multi-dimensional and irregular domain problems, are used in many papers to carry out the numerical solutions of PDEs. There are different kinds of meshless methods, like the weak forms element free Galerkin (EFG) method, the meshless local Petrov–Galerkin (MLPG) method and the MLPG based on particular solutions (MLPG-PS) [28–32], the meshless techniques based on collocation techniques [33,34], the method of approximate particular solutions [35], and the meshless techniques based on the combination of weak forms and collocation technique [36]. A recent review of the meshless methods was given in [37].

In the paper we are going to develop a new meshless method of domain type collocation techniques in terms of homogeneous boundary functions as the numerical bases [38,39]. Previously, the technique is used to solve the inverse geometry problem to recover a missing inner shape. The current numerical solution of the nonlinear inverse Cauchy problem is based on a novel scheme, by directly collocating points inside the doubly-connected domain and solving the transformed elliptic type equation iteratively to find the numerical solution, which is designed to automatically satisfy the over-specified Cauchy boundary conditions on the outer boundary.

The remainder of this paper is arranged as follows. In Section 2 we prescribe the nonlinear inverse Cauchy problem of the nonlinear elliptic equation in a doubly-connected plane domain and derive a homogenization function, which annihilates the over-specified Cauchy data on the outer boundary. Then, a homogenization technique and a new idea of boundary functions are introduced in Section 3, where we express the transformed variable in terms of the boundary functions as the bases, which automatically satisfy the homogeneous Cauchy boundary conditions on the outer boundary. In Section 4 we develop the numerical algorithms of the homogenization/boundary function method (HBFM). Seven numerical examples are given in Section 5 to assess the performance of the new HBFM in the recovery of different inner boundary data for different nonlinear elliptic equations. Finally, the conclusions are drawn in Section 6.

2. Variable transformation and homogenization technique

2.1. Nonlinear inverse Cauchy problem

Let Γ_o, a Lipschitz continuous and simple closed curve in the plane $\mathbb{R}^2$, be the boundary of a bounded domain $\Omega_o \subset \mathbb{R}^2$. We suppose that the zero point $(0, 0) \in \Omega_o$ and Ω_o is a star-like domain, which means that for each azimuth angle $\theta \in [0, 2\pi]$ as shown in Fig. 1 the ray emitting from the zero point $(0, 0)$ interacts with Γ_o at only one point. Similarly, Γ_i is a Lipschitz continuous and simple closed curve and is the boundary of a bounded domain $(0, 0) \in \Omega_i \subset \mathbb{R}^2$, which is also a star-like domain. Then we can form a doubly-connected domain $\Omega := \Omega_o/\Omega_i$ as shown in Fig. 1. Ω in the plane $\mathbb{R}^2$ being a doubly-connected domain means that any closed curve in Ω enclosed Ω_i cannot be continuously shrunk into a single point without leaving Ω.

We consider the following nonlinear inverse Cauchy problem to determine the unknown inner boundary data of the nonlinear elliptic type PDE:

$$\Delta u(x, y) + \lambda u(x, y) = H(x, y) + Q(u), \quad (x, y) \in \Omega \subset \mathbb{R}^2, \tag{1}$$

$$u = h(x, y), \quad (x, y) \in \Gamma_o, \tag{2}$$

$$u_n = g(x, y), \quad (x, y) \in \Gamma_o, \tag{3}$$

where $H(x, y)$, $h(x, y)$ and $g(x, y)$ are given bounded functions and $Q(u)$ is a nonlinear function of u satisfying the Lipschitz condition. The parameter λ in Eq. (1) may be positive, zero or negative. $u_n = \nabla u \cdot \mathbf{n}$ is a normal derivative of u on Γ_o with $\mathbf{n}$ a unit normal direction. Ω is a star-like doubly-connected domain with boundary $\Gamma = \Gamma_o \cup \Gamma_i$, where $\Gamma_o \cap \Gamma_i = \varnothing$. While Γ_o denotes an outer boundary, Γ_i is an inner boundary. We are looking for the inner boundary condition $f(x, y) = u(x, y),\ (x, y) \in \Gamma_i$, so that, the following forward problem is satisfied

$$\Delta u(x, y) + \lambda u(x, y) = H(x, y) + Q(u), \quad (x, y) \in \Omega \subset \mathbb{R}^2, \tag{4}$$

$$u = h(x, y), \quad (x, y) \in \Gamma_o, \tag{5}$$

$$u = f(x, y), \quad (x, y) \in \Gamma_i, \tag{6}$$

and respecting additional conditions $u_n = g(x, y),\ (x, y) \in \Gamma_o$.